Topology 550 Day 2

Outline
- Comparing topologies
- Closures and interiors
- Basis for a topology.

Announcements
- H.W. #1 Due Thursday of next week.

Comparing topologies

Given a set X with topologies τ_1 and τ_2, we say τ_1 is finer than τ_2 if $\tau_2 \subset \tau_1$

we say τ_2 is coarser than τ_1 if $\tau_2 \subset \tau_1$

we say τ_1 and τ_2 are incomparible if $\tau_1 \not\subset \tau_2$ and $\tau_2 \not\subset \tau_1$.

Ex $X = \{1, 2\}$ If $\tau_1 = \{\emptyset, \{1\}, \{1,2\}\}$ $\tau_I = \{\emptyset, \{1,2\}\}$

$\tau_2 = \{\emptyset, \{2\}, \{1,2\}\}$ $\tau_D = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

then τ_1 finer than τ_I

τ_1 and τ_2 incomparible

τ_1 coarser than τ_D.

Closures & Interiors

If (X, τ) is a top. space and $D \subset X$, then the interior of D, denoted $\text{int}(D)$, is the union of all open sets contained in D.

Moreover, the <u>closure</u> of D, denoted $\overline{D}$ is the intersection of all closed sets that contain D.

(Ex: Show $\overline{D}$ is always a closed set)
(Hint: Use the arbitray version of DeMorgan's Law $X\setminus(Y\cup Z) = (X\setminus Y)\cap(X\setminus Z)$.

<u>Ex</u> If $(X,\mathcal{T})$ is a top. space with the discrete topology and $A\subset X$.

$$\overline{A} = A \quad \text{and} \quad \mathrm{int}(A) = A.$$

<u>Basis for a topology</u>

Analogy:

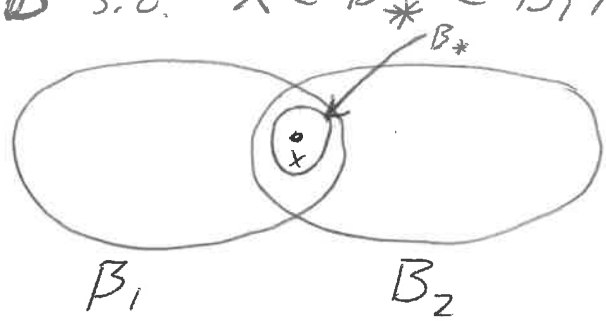
Linear algebra: Every vector is a linear combo. of basic vectors.

Topology: Every open set is a union of basic open sets.

<u>Def</u> Let X be a set. A <u>basis for a topology</u> on X is a set $\mathcal{B}\subset\mathcal{P}(X)$ s.t.

① $\forall x\in X$, $\exists B\in\mathcal{B}$ s.t. $x\in B$.

② If $B_1, B_2\in\mathcal{B}$ and $x\in B_1\cap B_2$, the $\exists B_*\in\mathcal{B}$ s.t. $x\in B_*\subset B_1\cap B_2$.



*We can use a basis $\mathcal{B}$ to define a topology. Declare $U \subset X$ to be open (in τ) if $\forall_{x \in U}$, $\exists B \in \mathcal{B}$ s.t. $x \in B \subset U$. We say τ is the <u>topology generated by $\mathcal{B}$</u>.

~~Exercise~~

Note: Every basis element is open.

Ex| $\mathbb{R}$ ~~with the standard topology~~ has basis $\mathcal{B} = \{$all open intervals$\}$

① $\forall x \in \mathbb{R}$ $\exists (x-1, x+1)$ s.t. $x \in (x-1, x+1)$.

② If (a,b) and (c,d) are elements of $\mathcal{B}$ and $x \in (a,b) \cap (c,d)$, the $x \in (c,b) \subset (a,b) \cap (c,d)$.

Hence $\mathcal{B}$ is a basis.

- The topology on $\mathbb{R}$ generated by $\mathcal{B}$ is called the standard topology.

Ex| $\mathbb{R}^2$ has basis $\mathcal{B} = \{$ interiors of all circles$\}$.

Ex| $\mathbb{R}^2$ has basis $\mathcal{B} = \{$interiors of all rectangles with sides parallel to the axes$\}$

* Later we with show that these basis generate the same topology.

* $\mathbb{R}_\ell$ and $\mathbb{R}_K$

Lemma 1: The topology $\mathcal{T}$ generated by a basis $\mathcal{B}$ is in fact a topology for X.

Proof: Claim 1: $\emptyset$ is open

Since $\emptyset$ contains no elements it is open vacuously.

Claim 2: X is open

Let $x \in X$. By def of basis $\exists B \in \mathcal{B}$ s.t. $x \in B \subset X$.

So, X is open.

Claim 3: Let $\{\mathcal{U}_\alpha\}$ be any collection of open sets, then $\bigcup_{\alpha \in A} \mathcal{U}_\alpha$ is open.

Let $\mathcal{U} = \bigcup_\alpha \mathcal{U}_\alpha$. Let $x \in \bigcup_\alpha \mathcal{U}_\alpha$. $x \in \mathcal{U}_\beta$ for some fixed β. Since $\mathcal{U}_\beta$ is open, $\exists B \in \mathcal{B}$ s.t. $x \in B \subset \mathcal{U}_\beta$. Hence, $x \in B \subset \bigcup_\alpha \mathcal{U}_\alpha$ and $\bigcup_\alpha \mathcal{U}_\alpha$ is open.

Claim 4: Let $\{\mathcal{U}_i\}_{i=1}^n$ be a collection of open sets, then $\bigcap_{i=1}^n \mathcal{U}_i$ is open.

If $\bigcap_{i=1}^n \mathcal{U}_i = \emptyset$, then we are done. Let $x \in \bigcap_{i=1}^n \mathcal{U}_i$. By definition of open, $\exists B_i \in \mathcal{B}$ s.t. $x \in B_i \subset \mathcal{U}_i$ for $1 \le i \le n$.

If $n=2$, by def of basis $\exists B_{1,2} \in \mathcal{B}$ s.t. $x \in B_{1,2} \subset B_1 \cap B_2$. Hence $\bigcap_{i=1}^n \mathcal{U}_i$ is open.

Suppose $\bigcap_{i=1}^n \mathcal{U}_i$ is open for $n=k$.

Suppose $n = k+1$. Since $\bigcap_{i=1}^k \mathcal{U}_i$ is open, $\exists B_* \in \mathcal{B}$ s.t. $x \in \mathcal{B}_* \subset \bigcap_{i=1}^k \mathcal{U}_i$. By defn of Basis $\exists B \in \mathcal{B}$ s.t. $x \in B \subset B_* \cap B_{k+1} \subset \bigcap_{i=1}^n \mathcal{U}_i$

Hence, by induction, $\bigcap_{i=1}^{n} \mathcal{U}_i$ is open.

By claims 1, 2, 3, 4 $\mathcal{T}$ is infact a topology. $\square$

Lemma 2: Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{U} \subset X$ is open iff $\mathcal{U}$ is the union of elements of $\mathcal{B}$.

Pf] Suppose $\mathcal{U}$ is open. By def. of open, $\forall x \in \mathcal{U}$ $\exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset \mathcal{U}$.

Claim: $\mathcal{U} = \bigcup_{x \in \mathcal{U}} B_x$

Let $x \in \mathcal{U}$, then $x \in B_x \subset \bigcup_{x \in \mathcal{U}} B_x$. So, $\mathcal{U} \subset \bigcup_{x \in \mathcal{U}} B_x$

Let $x \in \bigcup_{x \in \mathcal{U}} B_x$, then $x \in B_x \subset \mathcal{U}$. So, $\bigcup_{x \in \mathcal{U}} B_x \subset \mathcal{U}$.

Next, suppose $\mathcal{U} = \bigcup_{A \in \mathcal{A}} B_A$ where $B_A \in \mathcal{B}$.

Let $x \in \mathcal{U} = \bigcup_{A \in \mathcal{A}} B_A$. Hence, $x \in B_A \subset \mathcal{U}$ for some $A \in \mathcal{A}$.

Hence, $\mathcal{U}$ is open by definition. $\square$

Lemma 3: Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{C}$ be a collection of open sets s.t.

(*) Given $U \subset X$ open and $x \in U$, $\exists\, C \in \mathcal{C}$ s.t. $x \in C \subset U$.

Then $\mathcal{C}$ is a basis for the topology $\mathcal{T}$.

Proof | Step 1: Show $\mathcal{C}$ is a basis for some topology $\mathcal{T}'$.

① Let $x \in X$, since X is open, then, by (*), $\exists\, C \in \mathcal{C}$ s.t. $x \in C \subset \mathcal{C}$.

② Let $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{C}$.

Since B_1 and B_2 are open by hypothesis, then $B_1 \cap B_2$ is open. By (*) $\exists\, C \in \mathcal{C}$ s.t. $x \in C \subset B_1 \cap B_2$.

Hence, by ① and ② $\mathcal{C}$ is a basis.

Step 2: Show $\mathcal{T} = \mathcal{T}'$

$\subseteq$ | Let $U \in \mathcal{T}$. Let $x \in U$. By (*), $\exists\, C \in \mathcal{C}$ s.t. $x \in C \subset U$. By def. of open w.r.t. a basis U is open in $\mathcal{T}'$, so $U \in \mathcal{T}'$.

$\supseteq$ | Let $U \in \mathcal{T}'$. By def. of open w.r.t a basis $\forall x \in U\ \exists\, C_x \in \mathcal{C}$ s.t. $x \in C_x \subset U$.

Hence, $U = \bigcup_{x \in U} C_x$ (the union of open sets in $\mathcal{T}$).

Thus, $U \in \mathcal{T}$.

So, $\mathcal{T} = \mathcal{T}'$ $\square$

Lemma 4 Let X be a set and $\mathcal{B}, \mathcal{B}'$ be bases for topologies τ and τ'. T.F.A.E.

1) $\tau' \supset \tau$ (τ' is finer than τ)

2) Given any $x \in X$ and $B \in \mathcal{B}$ s.t. $x \in B$, $\exists\, B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.

Pf. In Munkres.

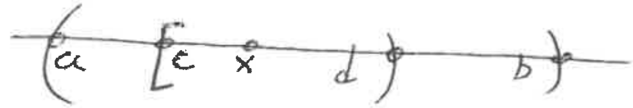
Applications

Let τ' be a topology on $\mathbb{R}$ induced by basis $\mathcal{B}' = \{[a,b) : a<b\}$. $\mathbb{R}_\ell = (\mathbb{R}, \tau')$ is the lower limit topology.

Claim: $\mathbb{R}_\ell$ is finer than $\mathbb{R}$.

Pf. Let $x \in \mathbb{R}$ and $(a,b) \in \mathcal{B}$ s.t. $x \in (a,b)$.

$$x \in \left[\cancel{a+}\,\frac{x+a}{2}, \cancel{b+}\,\frac{x+b}{2}\right) =: [c,d) \subset (a,b)$$



Hence, by Lemma 4, $\mathbb{R}_\ell$ is finer than $\mathbb{R}$. (In fact it is strictly finer).

Ex| Let $K = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$

$\mathcal{B}'' = \{(a,b) \mid a<b\} \cup \{(a,b)-K \mid a<b\}$

is a basis for $\mathbb{R}$ with topollogy τ''.

Denote $\mathbb{R}_K = (\mathbb{R}, \tau'')$

Q: How does $\mathbb{R}_\ell, \mathbb{R}, \mathbb{R}_K$ all compare?