Topology 550 Day 1 F17

Outline – Syllabus & name game.
- Set theory basics
- Def. of top. space
- Examples

Announcements
- H.W. #1 posted to course webpage. Due Sept 7th
- OH: Tu 5-6pm.

Sets & operations

A <u>set</u> is a collection of objects called <u>elements</u>

Notation	Meaning
$x \in X$	x is an element of X.
$X \subset Y$	every element of X is an element of Y
$X = Y$	$X \subset Y$ and $Y \subset X$
$\emptyset$	the empty set

- $X \cup Y = \{a \mid a \in X \text{ \underline{or} } a \in Y\}$
- $X \cap Y = \{a \mid a \in X \text{ \underline{and} } a \in Y\}$
- $X \setminus Y = X - Y = \{a \mid a \in X \text{ and } a \notin Y\}$
- $X \times Y = \{(a,b) \mid a \in X \text{ and } b \in Y\}$

Ex $\mathbb{R}$ = real line

$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ = cartesian plane.

- $\mathbb{P}(X) = \{A \mid A \subset X\}$.

Ex $\mathbb{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.

Arbitrary unions and intersections

Let $\mathcal{A}$ denote a collection of sets.

$$\bigcup_{A\in\mathcal{A}} A = \{a \mid a\in A \text{ \underline{for some} } A\in\mathcal{A}\}$$

$$\bigcap_{A\in\mathcal{A}} A = \{a \mid a\in A \text{ \underline{for all} } A\in\mathcal{A}\}$$

Ex: positive integers

$$\mathcal{A} = \{(-\tfrac{1}{n}, \tfrac{1}{n}) \mid n\in\mathbb{Z}^+\} \subset \mathcal{P}(\mathbb{R})$$

$$\bigcup_{A\in\mathcal{A}} A = (-1,1) \qquad \bigcap_{A\in\mathcal{A}} A = \{0\}$$

Rules of Set theory

Distributive Laws: If X, Y and Z are sets

$$X\cap(Y\cup Z) = (X\cap Y)\cup(X\cap Z)$$

$$X\cup(Y\cap Z) = (X\cup Y)\cap(X\cup Z)$$

De Morgan's Laws:

$$X-(Y\cup Z) = (X-Y)\cap(X-Z)$$

$$X-(Y\cap Z) = (X-Y)\cup(X-Z)$$

Functions

- Given sets X and Y, a function $f: X \to Y$ associates to each $x \in X$ exactly one element of Y, denoted $f(x)$.
- Given a function $f: X \to Y$ and $A \subset X$, the restriction of f to A is a function
  $$f|_A : A \to Y \quad \text{s.t.} \quad f|_A(a) = f(a) \text{ for all } a \in A.$$
- Given functions $f: X \to Y$ and $g: Y \to Z$, the composition $(g \circ f): X \to Y$ is defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$.
- f is injective or one-to-one or 1-to-1 if $x, x' \in X$ s.t. $x \neq x'$ implies $f(x) \neq f(x')$.
- f is surjective or onto if $\forall y \in Y$, $\exists x \in X$ s.t. $f(x) = y$.
- f is bijective if it is both injective and surjective.
- If $A \subset X$ and $f: X \to Y$, then
  $$f(A) = \{ y \in Y \mid y = f(x) \text{ for some } x \in X \}.$$
  Moreover, if $B \subset Y$
  $$f^{-1}(B) = \{ x \in X \mid f(x) \in B \}.$$

Rules for pre images

If $f: X \to Y$, $A \subset X$ and $B \subset Y$.

$A \subset f^{-1}(f(A))$ equality holds if f is inj.

$f(f^{-1}(B)) \subset B$ equality holds if f is surj.

* Def. of topological space

Def A *topology* on a set X is $\tau \subset \mathbb{P}(X)$ s.t.

1) Any arbitrary union of elements of τ is an element of τ. (closed under unions)
2) Any *finite* intersection of elements of τ is an element of τ. (closed under *finite* intersections)
3) $\emptyset \in \tau$ and $X \in \tau$

The pair (X, τ) is a *topological space*.

Given a top. space (X, τ), $U \subset X$ is *open* if $U \in \tau$. $U \subset X$ is *closed* if $X \setminus U \in \tau$.

Ex $\{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}\} \subset \mathbb{P}(\{1,2,3\})$ is a topology on $\{1,2,3\}$

Ex $\{\emptyset, X\}$ is a topology on X called the *indescrete* topology

Ex | $\mathbb{P}(X)$ is a topology on X called the discrete topology.

Pf | Let $\tau = \mathbb{P}(X)$. We want to show τ is a topology on X.

1) Let $\bigcup_{A \in \mathcal{A}} A$ be an arbitrary union $\mathcal{A} \subset \mathbb{P}(X) = \tau$.

Since $\forall A \in \mathcal{A}$, $A \subset X$, then $\bigcup_{A \in \mathcal{A}} A \subset X$.

Hence, $\bigcup_{A \in \mathcal{A}} A \in \mathbb{P}(X) = \tau$. ✓

2) Let $\bigcap_{i=1}^{n} A_i$ be a finite intersection s.t. $A_i \subset X$ for all $1 \leq i \leq n$.

Since $\forall_{i \in \{1, \ldots, n\}}$ $A_i \subset X$, then $\bigcap_{i=1}^{n} A_i \subset X$.

Hence, $\bigcap_{i=1}^{n} A_i \in \mathbb{P}(X) = \tau$. ✓

3) By def. of power set, $\emptyset, X \in \mathbb{P}(X) = \tau$.

~~Hence~~ Since 1), 2) and 3) above hold, then $\mathbb{P}(X)$ is a topology on X.