Announcements

- HW due Today (leave at 6:30pm)
- All projects accounted for.

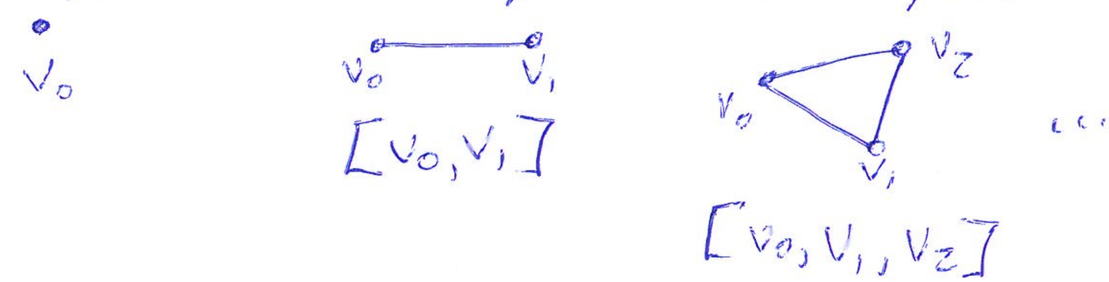
Outline

- Review of Δ-complex
- Simplicial Homology.

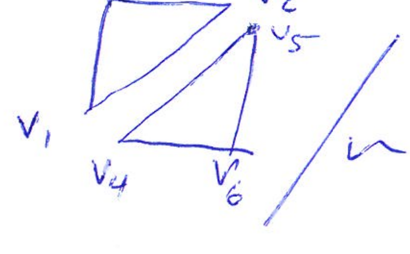
Review

An <u>n-simplex</u> is the smallest "non-degenerate" convex set containing $n+1$ ~~points~~ <u>ordered</u> points

0-simplex 1-simplex 2-simplex



Def A <u>Δ-complex</u> is the quotient space of a collection of disjiont simplicies obtained by identifying certain faces via cononical linear homeomorphisms that <u>preserve the ordering of vertices</u>.



Given a Δ-complex X that is the quotient of simplices including the n-simplex Δ^n_α, then there is a natural inclusion map $\sigma_\alpha : \Delta^n \to X$. Define e^n_α to be the interior of Δ^n_α. Then $\sigma_\alpha|_{e^n_\alpha}$ is a homeomorphism by definition of Δ-complex.

<u>Simplicial Homology</u>

Let X be a Δ-complex.

Let $\Delta_n(X)$ be the free abelian group generated by ~~all n-sim the open n-simplices e^n_α of X~~ σ_α (the inclusion maps for each Δ^n_α in X).

Note: Elements of $\Delta_n(X)$ are called <u>n-chains</u> and can be written as $\sum_{\alpha\in A} n_\alpha \sigma_\alpha$ where $n_\alpha \in \mathbb{Z}$ and $\{e^n_\alpha\}_{\alpha\in A}$ is the collection of all open n-simpleces in X.

<u>Def</u> The boundary of an n-simplex is defined as

$$\partial([v_0, \dots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots v_n]$$

This notation means remove!

Ex| $\partial([v_0, v_1, v_2]) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$

We can extend ∂ to a homomorphism from $\Delta_n(X)$ to $\Delta_{n-1}(X)$ by defining ∂ on each generator $\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$

$$\partial(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha |_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

Note| ∂_n is well-defined since LHS $\in \Delta_{n-1}(X)$.

Ex| Check that ∂ is a homomorphism.

Lemma 2.1| The composition $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}$ is zero.

Pf| $\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma |_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$

$$\partial_{n-1}(\partial_n(\sigma)) = \sum_{j=0}^{n-1} (-1)^j \sum_{i=j+1}^{n} (-1)^i [v_0, \dots \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]$$
$$+ \sum_{j=0}^{n-1} (-1)^{j-1} \sum_{i=0}^{j-1} (-1)^i [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots v_n]$$
$$= \sum \left((-1)^j (-1)^i + (-1)^{j-1} (-1)^i\right) [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots v_n]$$
$$= 0$$

Ex

$$\partial_{n-1}\partial_n([v_0 v_1 v_2 v_3])$$

$$\partial_{n-1}([v_1 v_2 v_3] - [v_0 v_2 v_3] + [v_0 v_1 v_3] - [v_0 v_1 v_2])$$

$$= \cancel{[v_2 v_3]} - \cancel{[v_1 v_3]} + \cancel{[v_1 v_2]}$$

$$- (\cancel{[v_2 v_3]} - \cancel{[v_0 v_3]} + \cancel{[v_0 v_2]})$$

$$+ (\cancel{[v_1 v_3]} - \cancel{[v_0 v_3]} + \cancel{[v_0 v_1]})$$

$$- (\cancel{[v_1 v_2]} - \cancel{[v_0 v_2]} + \cancel{[v_0 v_1]})$$

$$= 0$$

Def In algebra, when we have a sequence of abelian groups and homomorphisms given by $\to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \to \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$

s.t. $\partial_n \partial_{n+1} = 0$ for all n, we call it a

Chain complex

Given any chain complex, since $\partial_n \circ \partial_{n+1} = 0$ we know $\mathrm{Im}(\partial_{n+1}) \subset \mathrm{Ker}(\partial_n)$.

We define the <u>nth homology group</u>

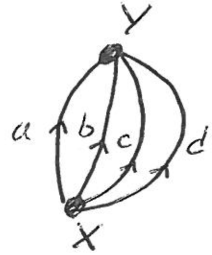
$$H_n = \mathrm{Ker}(\partial_n) / \mathrm{Im}(\partial_{n+1})$$

Given a Δ-complex X, form the chain complex $\to \Delta_{n+1}(X) \to \Delta_n(X) \to \cdots \to \Delta_0(X) \to 0$

The n-th homology group of X is

$$H_n(X) = \mathrm{Ker}(\partial_n) / \mathrm{Im}(\partial_{n+1}).$$

<u>Ex</u> Calculate $H_n($  $)$



$$\underset{0}{\overset{\|}{\Delta_{\geq 2}(X)}} \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0 \xrightarrow{\partial_0} \underset{0}{\overset{\|}{\Delta_{-1}}}$$

$a \to y - x$ $\quad y$

$b \to y - x$ $\quad x$

$c \to y - x$

$d \to y - x$

$\mathrm{Ker}(\partial_1)$ generated by $a-b,\ b-c,\ c-d$

$\mathrm{Im}(\partial_2) = 0$

$H_1(X) = \mathbb{Z}^3$