Announcements
- HW due today (due at 6:30 pm)
- All projects accounted for.

Outline
- Review of $\Delta$-complex
- Simplicial Homology

Review
An $n$-simplex is the smallest non-degenerate convex set containing $n+1$ points ordered points.

0-simplex
\[ V_0 \]

1-simplex
\[ V_0 - V_1 \]
\[ \left[ V_0, V_1 \right] \]

2-simplex
\[ V_0 \]
\[ V_1 \]
\[ V_2 \]
\[ \left[ V_0, V_1, V_2 \right] \]

Def A $\Delta$-complex is the quotient space of a collection of disjoint simplicies obtained by identifying certain faces via canonical linear homeomorphisms that preserve the ordering of vertices.
Given a $\Delta$-complex $X$ that is the quotient of simplices in clading the $n$-simplex $\Delta_n$, then there is a natural inclusion map $\sigma^* : \Delta^n \rightarrow X$. Define $E^*_n$ to be the interior of $\Delta_n$. Then $\sigma^* / E^*_n$ is a homeomorphism by definition of $\Delta$-complex.

Simplicial Homology

Let $X$ be a $\Delta$-complex.

Let $\Delta_n(X)$ be the free abelian group generated by all open $n$-simplices $E^*_n$ of $X$ (the inclusion maps for each $\Delta$ in $X$).

Note: Elements of $\Delta_n(X)$ are called $n$-chains and can be written as $\sum_{\sigma \in \Delta} n_{\sigma} \sigma^*$, where $n_{\sigma} \in \mathbb{Z}$ and $\sum E^*_n$ is the collection of all open $n$-simplices in $X$.

**Def:** The boundary of an $n$-simplex is defined as

$$\partial([v_0, \ldots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

This notation means remove $v_i$. 


Example
\[ \partial([v_0, v_1, v_2]) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \]

We can extend \( \partial \) to a homomorphism from \( \Delta_n(x) \) to \( \Delta_{n-1}(x) \) by defining \( \partial \) on each generator \( \partial_n : \Delta_n(x) \rightarrow \Delta_{n-1}(x) \)

\[ \partial(\sigma) = \sum_{i=0}^{n} (-1)^i \left[ v_0, \ldots, \hat{v}_i, \ldots, v_n \right] \]

Note: \( \partial_n \) is well-defined since LHS \( \in \Delta_{n-1}(x) \).

Example Check that \( \partial \) is a homomorphism.

Lemma 2.11 The composition \( \Delta_n(x) \xrightarrow{\partial_n} \Delta_{n-1}(x) \xrightarrow{\partial_{n-1}} \Delta_{n-2} \)

is zero.

Proof
\[ \partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \left[ v_0, \ldots, \hat{v}_i, \ldots, v_n \right] \]

\[ \partial_{n-1}(\partial_n(\sigma)) = \sum_{j=0}^{n-1} (-1)^j \left[ v_0, \ldots, \hat{v}_j, \ldots, v_n \right] \]

\[ + \sum_{j=0}^{n-1} (-1)^{j-1} \sum_{i=0}^{n-1} (-1)^i \left[ v_0, \ldots, \hat{v}_i, \ldots, v_n \right] \]

\[ = \sum ((-1)^j (-1)^i + (-1)^{j-1} (-1)^i) \left[ v_0, \ldots, \hat{v}_j, \ldots, v_n \right] \]

\[ = 0 \]
\[
E_x^1 \quad \partial_n (\mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_3) \\
\partial^{n-1} (\mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3) - \mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_3) + \mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_3) - \mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_3) = 0
\]

**Def:** In algebra, when we have a sequence of abelian groups and homomorphisms given by \( \rightarrow C_{n+1} \xrightarrow{\partial_n} C_n \xrightarrow{\partial_2} C_0 \xrightarrow{0} 0 \) such that \( \partial_n \partial_{n+1} = 0 \) for all \( n \), we call it a **Chain complex**
Given any chain complex, since 
\[ \partial_n \circ \partial_{n+1} = 0 \] we know \( \text{Im}(\partial_{n+1}) \subset \text{Ker}(\partial_n) \). 
we define the \( n \)-th homology group
\[ H_n = \frac{\text{Ker}(\partial_n)}{\text{Im}(\partial_{n+1})} \]

Given a \( \Delta \)-complex \( X \), form the chain complex 
\[ \Delta_{n+1}(X) \to \Delta_n(X) \to \cdots \to \Delta_0(X) \to 0 \]
The \( n \)-th homology group of \( X \) is 
\[ H_n(X) = \frac{\text{Ker}(\partial_n)}{\text{Im}(\partial_{n+1})} \]

**Example Calculate** 
\[ H_n(\emptyset) \]

\[ \Delta \geq z(X) \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0 \xrightarrow{\partial_0} \Delta_{-1} \]

\[ \begin{array}{c}
0 \\
\begin{array}{c}
a \\
b \\
c \\
d \\
\end{array} \xrightarrow{y-x} \begin{array}{c}
0 \\
y \\
x \\
y-x \\
y-x \\
\end{array} \\
\text{Ker}(\partial_1) \text{ generated by } a-b, b-c, c-d \\
\text{Im}(\partial_2) = 0 \\
\end{array} \]

\[ H_1(X) = \mathbb{Z}^3 \]