Announcements

- HW 5 Due today
- Midterm 2 partial in class partial take home 3 in class questions infasizing HW problems and proofs presented in class. 2 (difficult) take home problems under HW rules.

Outline

- Review of big homotopy result
- Free products of groups and ~~generator~~ presentations of groups.

Def| Let G and H be groups. The product group $G \times H$ is the set $\{(g,h) \mid g \in G \text{ and } h \in H\}$ with the binary operation $(g_1, h_1) * (g_2, h_2) = (g_1 * g_2, h_1 * h_2)$.

The problem| This definition gives rise to a weird bit of commutivity $(g, 1_H) * (1_G, h) = (g, h) = (1_G, h) * (g, 1_H)$.

This seems like an unussual relation to have especially for non comutative groups.

Def| Let G and H be groups. free product of G and H, denoted $G * H$ is the set of all words
$g_1 h_1 g_2 h_2 \cdots g_m h_m$
$g_1 h_1 \cdots h_{m-1} g_m$
$h_1 g_1 \cdots h_m g_m$
$h_1 g_1 \cdots g_{m-1} h_m$

of arbitrary finite length ~~where $g_i \in G$, $h_i \in H$~~

where $g_i \in G - \{1\}$ for each i and

$h_i \in H - \{1\}$ for each j.

under the operation of juxtaposition

Given $(a_1 b_1 a_2 b_2 \cdots a_m b_m) * c_1 d_1, \cdots c_{n-1} d_n = a_1 b_1 \cdots a_m b_m c_1 d_1 \cdots c_{n-1} d_n$

Group axioms

Idetity: $G*H$ is defined to contain the empty word. $\emptyset$

Closure: Two words justaposed always give a word and after we reduce the word we get an element of $G*H$

Ex| $a \in G$ and $b \in H$

$b^{-1}a^{-1}, ab \in G*H$

$$abb^{-1}a^{-1} \notin G*H$$
$$\shortparallel$$
$$aa^{-1} \notin G*H$$
$$\shortparallel$$
$$\emptyset \in G*H$$

Inverse element:

$$a_1 b_1 a_2 b_2 \cdots a_n b_m * b_m^{-1} a_n^{-1} \cdots b_1^{-1} a_1^{-1} = \emptyset$$

Associativity: Too hard for us!

Examples

$$\mathbb{Z} * \mathbb{Z} \cong \langle a \rangle * \langle b \rangle$$

$$aba^2b^3a^{-1}ba^{-2} \in \mathbb{Z} * \mathbb{Z}$$

This is the free group on two generators.

$$\mathbb{Z}_2 * \mathbb{Z}_2 \cong \langle a \mid a^2 = 1 \rangle * \langle b \mid b^2 = 1 \rangle$$

$$ababa,\ ab \in \mathbb{Z}_2 * \mathbb{Z}_2$$

Def The free group on n generators is defined to be $\underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{n\text{-times.}}$.

Universal property of free products

Given Groups G_1, G_2 and H and homomorphisms $\varphi_1 : G_1 \to H$ and $\varphi_2 : G_2 \to H$, there exists a <u>unique</u> homomorphism $\varphi : G_1 * G_2 \to H$ s.t.

for every one leter word $g \in G_1 * G_2$ s.t. $g \in G_1$ $\varphi(g) = \varphi_1(g)$

" " " " " " $g \in G_2$ $\varphi(g) = \varphi_2(g)$.

Namely, $\varphi(a_1 b_1 a_2 b_2 \cdots a_m b_m) = \varphi_1(a_1)\varphi_2(b_1)\varphi_1(a_2) \cdots \varphi_2(b_m)$.

In particular, the inclutions $G \hookrightarrow G \times H$ and $H \hookrightarrow G \times H$ induce a unique (surjective) homomorphism $G * H \to G \times H$.

Def| Given a group $(G, *)$ a subset of G denoted by $S \subset G$ is a *generating set* for G if every element in G can be expressed as a combination of elements in S and their inverses under the operation $*$.

Ex| What is a generating set for

$$\mathbb{Z} \oplus \mathbb{Z} \quad \{(0,1), (1,0)\}$$

what is a generating set for S_4
(the semetric group on 4 letters)

$$\{(1\,2), (2,3), (3,4)\}$$

Def| A group presentation $\langle S \mid R \rangle$ is a set of generator S and a set of reduced words in S and the inverses of elements in S.

As a group $\langle S \mid R \rangle \cong F_S / \langle\langle R \rangle\rangle$

where F_S is the free group on S and $\langle\langle R \rangle\rangle$ is the normal subgroup of F_S generated by elements of R.

Examples

$$\mathbb{Z}_2 \cong \mathbb{Z}/2\mathbb{Z} \cong \langle a \mid a^2 \rangle$$

$$\mathbb{Z}_n \cong \langle a \mid a^n \rangle$$

$$\mathbb{Z} \cong \langle a \mid \ \rangle$$

$$\mathbb{Z} \oplus \mathbb{Z}_n \cong \langle a, b \mid b^n, aba^{-1}b^{-1} \rangle$$

$$\mathbb{Z} * \mathbb{Z} \cong \langle a, b \mid \ \rangle$$

$$\begin{aligned} S_4 \cong \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^2, \sigma_2^2, \sigma_3^2, &\ \sigma_1\sigma_3\sigma_1^{-1}\sigma_3^{-1} \\ &\ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \\ &\ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \rangle \end{aligned}$$

$$\begin{aligned} B_4 = \text{braid group on 4 strands} \cong \langle \sigma_1, \sigma_2, \sigma_3 \mid &\ \sigma_1\sigma_3\sigma_1^{-1}\sigma_3^{-1} \\ &\ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \\ &\ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \rangle \end{aligned}$$

$$\mathbb{Z}_2 * \mathbb{Z}_2 \cong \langle a, b \mid a^2 = 1, b^2 = 1 \rangle \cong \frac{\mathbb{Z}_{\langle a \rangle} * \mathbb{Z}_{\langle b \rangle}}{\langle\langle a^2, b^2 \rangle\rangle}$$

$$\langle a, b \mid ab^2 \rangle * \langle c, d \mid c^2, d^2 \rangle = \langle a, b, c, d \mid ab^2, c^2, d^2 \rangle$$

Announcements

- New HW due Tues of Next week
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Out line

- Review of group persentations
- Van Kampen's theorem.

Review

A free product of groups G and H, denoted $G*H$ is the group of reduced words of finite length under the group operation of juxtaposition

i.e. $\underset{\langle a\rangle}{\mathbb{Z}} * \underset{\langle b\rangle}{\mathbb{Z}} \quad (a^2b^{-1}a)*(bab^{-4}) = a^2b^{-1}abab^{-4}$

A group presentation $\langle S|R\rangle$ is a set of generators S and a set of words in S (and the inverses of these elements) R.

$$\langle S|R\rangle \cong F_S/\langle\langle R\rangle\rangle$$

i.e. $\langle a,b \mid a^2\rangle \cong \mathbb{Z}_2 * \mathbb{Z}$

$\langle a,b,c \mid a^3, b^2, c^2, aca^{-1}c^{-1}\rangle \cong \mathbb{Z}_3 * (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$

A basic property of the free product $G * H$

Given homomorphisms $\varphi_g : G \to K$ and $\varphi_h : H \to K$

There exists a unique homomorphism

$$\varphi : G * H \to K \quad \text{s.t.}$$

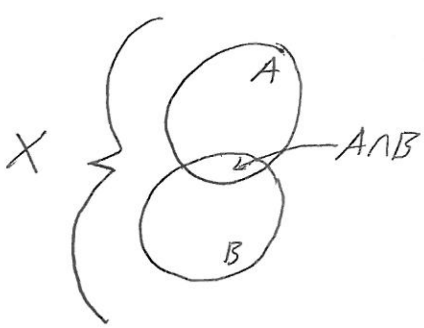
$\varphi(g) = \varphi_g(g)$ for all length one words $g \in G * H$ s.t. $g \in$

and $\varphi(h) = \varphi_h(h)$ for all length one words $h \in G * H$ s.t. $h \in H$.

Namely $\varphi(g_1 h_1 g_2 h_2 \cdots g_m h_m) = \varphi_g(g_1) \cdot \varphi_h(h_1) \cdot \cdots \varphi_g(g_m) \cdot \varphi_h(h$

Set up for Van Kampen

Suppose X is the union of two path-connected open sets A and B s.t. $A \cap B$ is path connected.



We have inclusion maps and their induced homomorphisms

$i_A : A \to X \quad (i_A)_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$

$i_B : B \to X \quad (i_B)_* : \pi_1(B, b_0) \to \pi_1(X, b_0)$

Note, by the basic property of free products, there exists a unique map $\varphi : \pi_1(A, a_0) * \pi_1(B, b_0) \to \pi_1(X, b_0)$ that extends $(i_A)_*$ and $(i_B)_*$.

Van Kampen's theorem will give us sufficient conditions for φ to be onto. It also tells us what the kernel of φ is.

Recall from group theory.

If $\varphi: G \to H$ is an onto homomorphism, then $H \cong G/\mathrm{Ker}\,\varphi$.

Hence, Van Kampen's theorem gives us a way to calculate $\pi_1(X, x_0)$.

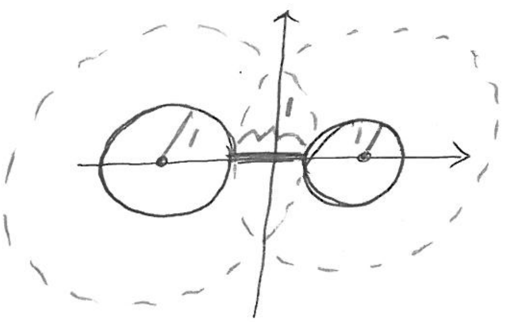
Thm (H: 1.20) If X is the union of path-connected open sets A and B s.t. $x_0 \in A \cap B$ and $A \cap B$ is path-connected, then $\varphi: \pi_1(A, x_0) * \pi_1(B, x_0) \to \pi_1(X, x_0)$ is onto. More over $\mathrm{Ker}(\varphi)$ is the normal subgroup generated by all elements of the form

$j_A(\omega) j_B(\omega)^{-1}$ where $j_A: \pi_1(A \cap B) \to \pi_1(A)$ and $j_B: \pi_1(A \cap B) \to \pi_1(B)$ are the induced maps from the inclusions $A \cap B \hookrightarrow A$ and $A \cap B \hookrightarrow B$.

Hence $\pi_1(X, x_0) \cong \dfrac{\pi_1(A, x_0) * \pi_1(B, x_0)}{\mathrm{Ker}(\varphi)}$.

Applications of Van Kampen's theorem

Ex] Let X be the eye glass graph.



$A = X \cap B_2((3/2,0))$

$B = X \cap B_2((-\frac{3}{2},0))$

So, $X = A \cup B$

Both A and B are open, path connected subsets.

Note $A \cap B = (-1/2, 1/2) \subset X\text{-axis}$

Since $A \cap B$ is convex, $\pi_1(A \cap B, \vec{0}) \cong \{1\}$

Hence j_A and j_B are both trivial maps.

By Van Kampen's $\pi_1(X, \vec{0}) \cong \dfrac{\pi_1(A,\vec{0}) * \pi_1(B,\vec{0})}{Ker(\varphi)}$

However, Both A and B deformation retract onto S^1

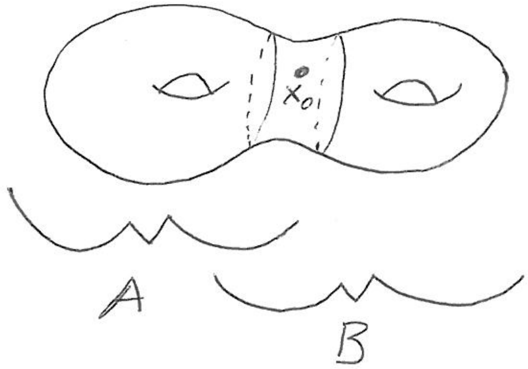
Check this as an exercise

Hence $\pi_1(A,\vec{0}) \cong \pi_1(B,\vec{0}) \cong \mathbb{Z}$

So, $\pi_1(X,\vec{0}) \cong \dfrac{\mathbb{Z} * \mathbb{Z}}{\langle\langle 1 \rangle\rangle} \cong \mathbb{Z} * \mathbb{Z} \cong F_2$

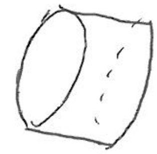
the free group on 2 variables.

<u>Ex</u> Let X be the genus 2 surface



$A =$  $\cong S^1 \times S^1 - D^2$

$B =$  $\cong S^2 \times S^1 - D^2$

$A \cap B =$  $\cong S^1 \times (0,1)$

Since A, B and $A \cap B$ are all open, path-connected subsets of X, we can apply Van Kampens theorem.

$$\pi_1(X, x_0) \cong \frac{\pi_1(A, x_0) * \pi_1(B, x_0)}{Ker(\varphi)}$$ Must find these.

Note: A and B deformation retract onto the wedge of two circles (ie $S^1 \vee S^1$ ∞)

Note: $S^1 \vee S^1$ is homotopic to the eyeglass graph

Proof: <u>Exercise</u>

Hence $\pi_1(A, x_0) \cong \pi_1(B, x_0) \cong \pi_1(\infty) \cong \pi_1(\text{O-O}) \cong F_2$

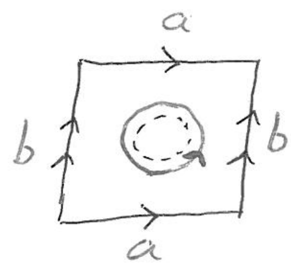
Since $S^1 \times (0,1)$ deformation retracts onto $S^1 \times \{1/2\}$, then $\pi_1(S^1 \times (0,1)) \cong \pi_1(A \cap B, x_0) \cong \mathbb{Z} \cong \langle w \rangle$

Recall the ~~induced~~ maps induced by the inclusion maps

$$j_A : \pi_1(A \cap B, x_0) \to \pi_1(A, x_0) \quad \text{and} \quad j_B : \pi_1(A \cap B, x_0) \to \pi_1(B, x_0)$$

$$j_A : \langle w \rangle \to \langle a, b \mid \rangle \quad \text{and} \quad j_B : \langle w \rangle \to \langle c, d \mid \rangle$$

We need to figue out what these maps do!



$$j_A(w) = aba^{-1}b^{-1}$$

$$j_B(w) = cdc^{-1}d^{-1}$$

So, $$\pi_1(X, x_0) \cong \frac{\langle a, b \mid \rangle * \langle c, d \mid \rangle}{\langle\langle aba^{-1}b^{-1}cdc^{-1}d^{-1}}$$

$$\cong \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$$

Problem (really a group theory problem)
Show the genus 2 surface is not homotopic to the torus $S^1 \times S^1$. (Hint: If two groups are isomorphic, then their abelianizations are isomorphic).