Homework Rules:

~~Cite~~

On each problem

- Cite if you have asked a faculty member for help.
- Cite if you have worked closely with another student.
- Cite if you have looked at any internet source.
- Always rewrite any solution that you had outside help on in your own words.
- Homeworks that do not follow these rules will recieve zeros.

Announcements

- HW due Thursday
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Outline

- Review homotopy equivalence
- Show that fundamental group is a homotopy invariant.

Review

Def If $f: X \to Y$ and $g: Y \to X$ are continuous maps s.t. $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$, then f and g are <u>homotopy equivalences</u> and X is homotopic to Y.

Ex S^1 is homotopic to $\mathbb{R}^2 - \{\vec{0}\}$.

Let $i: S^1 \to \mathbb{R}^2 - \{\vec{0}\}$ be the inclusion and let $r: \mathbb{R}^2 - \{\vec{0}\} \to S^1$ by $r(\vec{x}) = \frac{\vec{x}}{\|\vec{x}\|}$ be a retraction, then $i \circ r \simeq id_{S^1}$ and $r \circ i \simeq id_{\mathbb{R}^2 - \{0\}}$. (check this for yourself).

Lemma 58.4

Let $h, k : X \longrightarrow Y$ be continuous maps s.t. $h(x_0) = y_0$ and $k(x_0) = y_1$. If h is homotopic to k, then there is a path α in Y from y_0 to y_1 s.t. $k_* = \hat{\alpha} \circ h_*$

i.e. the following diagram commutes

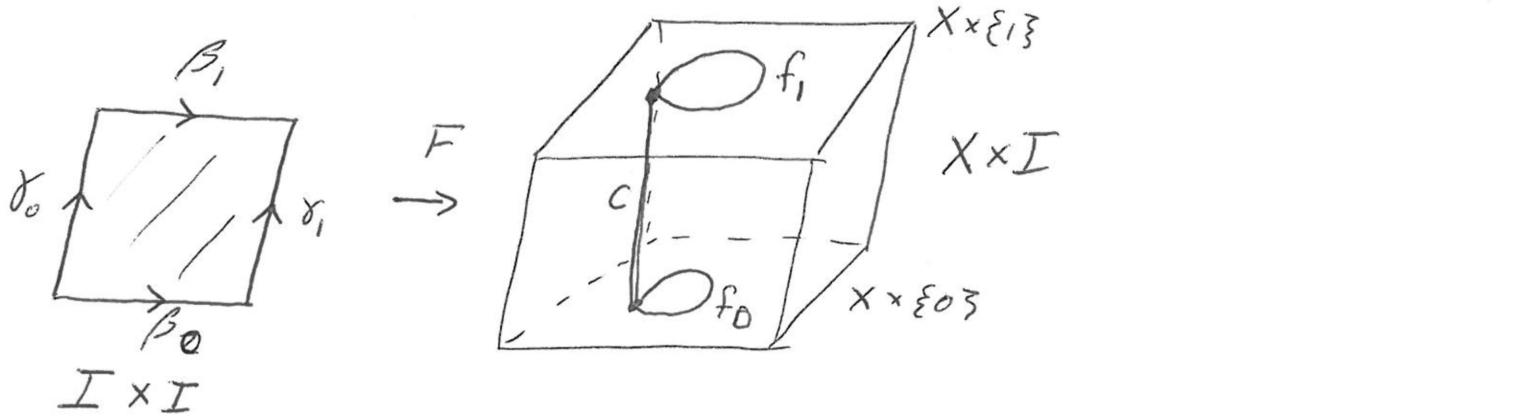
$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "h_*"] \arrow[dr, "k_*"'] & \pi_1(Y, y_0) \arrow[d, "\hat{\alpha}"] \\ & \pi_1(Y, y_1) \end{tikzcd}$$

Pf] Let f be a loop in X based at x_0.

WTS: $k_*([f]) = \hat{\alpha} \circ h_*([f])$

Consider $f_0(s) = (f(s), 0)$ and $f_1(s) = (f(s), 1)$ in $X \times I$

Let $c(t) = (x_0, t)$, a path in $X \times I$



Let $H : X \times I \to Y$ be the homotopy from h to k.

Note $H \circ f_0(s) = h \circ f(s)$ and $H \circ f_1(s) = k \circ f(s)$.

Call $H \circ c = \alpha : I \to Y$. from y_0 to y_1.

Define $F : I \times I \to X \times I$ by $F(s,t) = (f(s), t)$

So, $F \circ \beta_0 = f_0$, $F \circ \beta_1 = f_1$, $F \circ \gamma_0 = c$ and $F \circ \gamma_1 = c$.

Since $I \times I$ is convex, $\beta_1 * \gamma_0$ is path-homotopic to $\gamma_1 * \beta_0$ via a path-homotopy G.

Hence, $F \circ G$ is a path homotopy in $X \times I$ from $f_1 * c$ to $\cancel{f_0 * c_2}$ $c * f_0$.

Similarly, $H \circ (F \circ G)$ is a path homotopy from $(k \circ f) * \alpha$ to $\alpha * (h \circ f)$.

So
$$[k \circ f] * [\alpha] = [\alpha] * [h \circ f]$$
$$[\bar{\alpha}] * [k \circ f] * [\alpha] = [h \circ f]$$
$$\hat{\alpha}([k \circ f]) = [h \circ f]$$
$$\hat{\alpha} \circ k_*([f]) = h_*([f])$$

Cor Let $h, k : X \to Y$ be homotopic maps s.t. $h(x_0) = y_0$ and $k(x_0) = y_1$. If h_* is one-to-one, onto or trivial, then so is k_*.

Thm (58.7) Let $f : X \to Y$ be continuous s.t. $f(x_0) = y_0$. If f is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

Pf Let $g : Y \to X$ be the homotopy inverse of f.

Consider $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1)$

$\Downarrow$ applying the π_1 functor

$\pi_1(X, x_0) \xrightarrow{(f_{x_0})_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(X, x_1) \xrightarrow{(f_{x_1})_*} \pi_1(Y, y_1)$

By lemma (58.4), since $g \circ f \simeq id_X$, then there exists a path α in X s.t. $(g \circ f)_* = \hat{\alpha} \circ (id_X)_* = \hat{\alpha}$.

Since $\hat{\alpha}$ is an isomorphism, then $g_* \circ f_*$ is an iso.

Again, by lemma (58.4) $(f \circ g)_* = \hat{\beta} \circ (id_Y)_* = \hat{\beta}$.

Since $\hat{\beta}$ is an isomorphism, then $f_* \circ g_*$ is an isomorphism.

Hence, f_* is a one-to-one and onto homomorphism.

So, f_* is an isomorphism. $\square$.

Announcements

- HW5 Due today
- Midterm 2 partial in class partial take home
  3 inclass questions infasizing HW problems and proofs presented in class. 2 (difficult) take home problems under HW rules.

Outline

- Review of big homotopy result

Def| Let G and H be groups. The product group $G \times H$ is the set $\{(g,h) \mid g \in G \text{ and } h \in H\}$ with the binary operation $(g_1, h_1) * (g_2, h_2) = (g_1 * g_2, h_1 * h_2)$.

The problem| This definition gives rise to a weird bit of commutivity $(g, 1_H) * (1_G, h) = (g, h) = (1_G, h) * (g, 1_H)$

This seems like an unussual relation to have especially for non comutative groups.

Def| Let G and H be groups. free product of G and H, denoted $G * H$ is the set of all words

$$g_1 h_1 g_2 h_2 \cdots g_m h_m$$
$$g_1 h_1 \cdots h_{m-1} g_m$$
$$h_1 g_1 \cdots h_m g_m$$
$$h_1 g_1 \cdots g_{m-1} h_m$$

of arbitrary finite length ~~wher $g_i \in G$, $h_i \in H$~~

where $g_i \in G - \{1\}$ for each i and

$h_i \in H - \{1\}$ for each i.

under the operation of juxtaposition

Given $(a_1 b_1 a_2 b_2 \cdots a_m b_m) * c_1 d_1, \cdots c_{n-1} d_n = a_1 b_1 \cdots a_m b_m c_1 d_1 \cdots c_n$

Group axioms

Idetity: $G * H$ is defined to contain the empty word. $\emptyset$

Closure: Two words justaposed always give a word and after we reduce the word we get an element of $G * H$

Ex $a \in G$ and $b \in H$

$b^{-1}a^{-1}, ab \in G * H$

$$abb^{-1}a^{-1} \notin G * H$$

$$\parallel$$

$$aa^{-1} \notin G * H$$

$$\parallel$$

$$\cancel{1}$$

$$\emptyset \in G * H$$

Inverse element:

$$a_1 b_1 a_2 b_2 \cdots a_n b_m * b_m^{-1} a_n^{-1} \cdots b_1^{-1} a_1^{-1} = \emptyset$$

Associativity: Too hard for us!

Examples

$$\mathbb{Z} * \mathbb{Z} \cong \langle a \rangle * \langle b \rangle$$

$$aba^2b^3a^{-1}ba^{-2} \in \mathbb{Z} * \mathbb{Z}$$

This is the free group on two generators.

$$\mathbb{Z}_2 * \mathbb{Z}_2 \cong \langle a \mid a^2 = 1 \rangle * \langle b \mid b^2 = 1 \rangle$$

$$ababa,\ ab \in \mathbb{Z}_2 * \mathbb{Z}_2$$

Def The free group on n genorators is defined to be $\underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{n\text{-times}}$.

Universal property of free products

Given Groups G_1, G_2 and H and homomorphisms $\varphi_1 : G_1 \to H$ and $\varphi_2 : G_2 \to H$, there exists a <u>unique</u> homomorphism $\varphi : G_1 * G_2 \to H$ s.t.

for every one leter word $g \in G_1 * G_2$ s.t. $g \in G_1$ $\varphi(g) = \varphi_1(g)$

" " " " " $g \in G_2$ $\varphi(g) = \varphi_2(g)$.

Namely, $\varphi(a_1 b_1 a_2 b_2 \cdots a_m b_m) = \varphi_1(a_1)\varphi_2(b_1)\varphi_1(a_2)\cdots\varphi_2(b_m)$.

In particular, the inclutions $G \hookrightarrow G \times H$ and $H \hookrightarrow G \times H$ induce a unique (surjective) homomorphism $G * H \to G \times H$.

Def | Given a group $(G, *)$ a subset of G denoted by $S \subset G$ is a <u>generating</u> <u>set</u> for G if every element in G can be expressed as a combination of elements in S and their inverses under the operation $*$.

Ex | What is a generating set for

$$\mathbb{Z} \oplus \mathbb{Z} \quad \{(0,1), (1,0)\}$$

what is a generating set for S_4 (the semetric group on 4 letters)

$$\{(1\,2), (2,3), (3,4)\}$$

Def | A group presentation $\langle S \mid R \rangle$ is a set of generator S and a set of reduced words in S and the inverses of elements in S.

As a group $\langle S \mid R \rangle \cong F_S / \langle\langle R \rangle\rangle$

where F_S is the free group on S and $\langle\langle R \rangle\rangle$ is the normal subgroup of F_S generated by elements of R.

Examples

$$\mathbb{Z}_2 \cong \mathbb{Z}/2\mathbb{Z} \cong \langle a \mid a^2 \rangle$$

$$\mathbb{Z}_n \cong \langle a \mid a^n \rangle$$

$$\mathbb{Z} \cong \langle a \mid \ \rangle$$

$$\mathbb{Z} \oplus \mathbb{Z}_n \cong \langle a, b \mid b^n, aba^{-1}b^{-1} \rangle$$

$$\mathbb{Z} * \mathbb{Z} \cong \langle a, b \mid \ \rangle$$

$$S_4 \cong \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_1\sigma_3\sigma_1^{-1}\sigma_3^{-1}, \\ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \\ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \rangle$$

B_4 = braid group on 4 strands $\cong \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_3\sigma_1^{-1}\sigma_3^{-1}$

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

$$\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \rangle$$

$$\mathbb{Z}_2 * \mathbb{Z}_2 \cong \langle a, b \mid a^2 = 1, b^2 = 1 \rangle \cong \frac{\mathbb{Z}_{\langle a \rangle} * \mathbb{Z}_{\langle b \rangle}}{\langle\langle a^2, b^2 \rangle\rangle}$$