Homework Rules:

- Cite if you have asked a faculty member for help.
- Cite if you have worked closely with another student.
- Cite if you have looked at any internet source.
- Always rewrite any solution that you had outside help on in your own words.
- Homeworks that do not follow these rules will receive Zeros.
Announcements

- HW due Thursday

Outline

- Review homotopy equivalence
- Show that fundamental group is a homotopy invariant.

Review

**Def** If \( f : X \to Y \) and \( g : Y \to X \) are continuous maps s.t. \( f \circ g = \text{id}_Y \) and \( g \circ f = \text{id}_X \), then \( f \) and \( g \) are homotopy equivalences and \( X \) is homotopic to \( Y \).

**Ex** \( S^1 \) is homotopic to \( \mathbb{R}^2 - \{0, 0\} \).

Let \( i : S^1 \to \mathbb{R}^2 - \{0, 0\} \) be the inclusion and let \( r : \mathbb{R}^2 - \{0, 0\} \to S^1 \) by \( r(x) = \frac{x}{\|x\|} \) be a retraction, then \( i \circ r = \text{id}_{S^1} \) and \( r \circ i = \text{id}_{\mathbb{R}^2 - \{0, 0\}}. \) (check this for yourself).
Lemma 58.41

Let \( h, k : X \to Y \) be continuous maps s.t. \( h(x_0) = y_0 \) and \( k(x_0) = Y_1 \). If \( h \) is homotopic to \( k \), then there is a path \( \gamma \) in \( Y \) from \( y_0 \) to \( y_1 \) s.t. \( k_* = \tilde{\alpha} \circ h_* \)

i.e. the following diagram commutes

\[
\begin{array}{ccc}
\pi_1(X, x_0) & \xrightarrow{h_*} & \pi_1(Y, y_0) \\
\downarrow{k_*} & & \downarrow{\gamma} \\
\pi_1(Y, y_1) & & \\
\end{array}
\]

Proof (Pf)

Let \( f \) be a loop in \( X \) based at \( x_0 \).

WTS: \( k_*([f]) = \tilde{\alpha} \circ h_*([f]) \)

Consider \( f_0(s) = (f(s), 0) \) and \( f_1(s) = (f(s), 1) \) in \( X \times I \)

Let \( c(t) = (x_0, t) \), a path in \( X \times I \)

Let \( H : X \times I \to Y \) be the homotopy from \( h \) to \( k \).
Note $H \circ f_0(s) = h \circ f(s)$ and $H \circ f_1(s) = k \circ f(s)$.

Call $H \circ c = \alpha : I \to Y_0$ from $Y_0$ to $Y_1$.

Define $F : I \times I \to X \times I$ by $F(s, t) = (f(s), t)$

So, $F \circ \beta_0 = f_0$, $F \circ \beta_1 = f_1$, $F \circ c_0 = c$ and $F \circ c_1 = c$.

Since $I \times I$ is convex, $\beta_1 \ast \beta_0$ is path-homotopic to $\gamma_1 \ast \beta_0$ via a path-homotopy $G$.

Hence, $F \circ G$ is a path homotopy in $X \times I$ from $f_1 \ast c$ to $f_0 \ast c \ast f_0$.

Similarly, $H \circ (F \circ G)$ is a path homotopy from $(k \circ f) \ast \alpha$ to $\alpha \ast (h \circ f)$.

So

$$[k \circ f] \ast [\alpha] = [\alpha] \ast [h \circ f]$$

$$[\alpha] \ast [k \circ f] \ast [\alpha] = [h \circ f]$$

$$\tilde{\alpha}([k \circ f]) = [h \circ f]$$

$$\tilde{\alpha} \circ k_*([f]) = h_*([f])$$
Cor Let $h, k : X \to Y$ be homotopic maps s.t. $h(x_0) = y_0$ and $k(x_0) = y_1$. If $h_\ast$ is one-to-one, onto or trivial, then so is $k_\ast$.

Thm (580.7) Let $f : X \to Y$ be continuous s.t. $f(x_0) = y_0$. If $f$ is a homotopy equivalence, then $f_\ast : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

Pf Let $g : Y \to X$ be the homotopy inverse of $f$.

Consider $\quad \begin{array}{c} (X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1) \\ \downarrow \text{applying the } \pi_1 \text{ functor} \\ \pi_1(X, x_0) \xrightarrow{(f_\ast)_\ast} \pi_1(Y, y_0) \xrightarrow{g_\ast} \pi_1(X, x_1) \xrightarrow{(f_\ast)_\ast} \pi_1(Y, y_1) \end{array}$

By lemma (580.4), since $gof = id_X$, then there exists a path $\alpha$ in $X$ s.t. $(gof)_\ast = \alpha o (id_X)_\ast = \alpha$.

Since $\alpha$ is an isomorphism, then $g_\ast of_\ast$ is an iso. Again, by lemna (580.4) $(fog)_\ast = \beta o (id_X)_\ast = \beta$.

Since $\beta$ is an isomorphism, then $f_\ast fog_\ast$ is an isomorphism. Hence, $f_\ast$ is a one-to-one and onto homomorphism. So, $f_\ast$ is an isomorphism. $\square$.
Announcements
- HW5 Due today
- Midterm 2 partial in class partial take home
  3 in class questions focusing HW problems and
  proofs presented in class. 2 (difficult)
take home problems under HW rules.

Outline
- Review of big homotopy result

**Def** Let $G$ and $H$ be groups. The product group
$G \times H$ is the set \{$(g, h) | g \in G \text{ and } h \in H$\} with
the binary operation $(g_1, h_1) * (g_2, h_2) = (g_1 g_2, h_1 h_2)$.

**The problem** This definition gives rise to a weird bit
of commutivity $(g_1, h_1) * (1_g, h) = (g_1, h) = (1_g, h) * (g_1, h)$
This seems like an unusual relation to have especially
for non commutative groups.

**Def** Let $G$ and $H$ be free product of $G$ and $H$, denoted
$G * H$ is the set of all words $g_1 h_1 g_2 h_2 \ldots g_n h_m$
$g_1 h_1 \ldots h_m g_m$
$h_1 g_1 \ldots h_m g_m$
$h_1 g_1 \ldots g_m h_m$

of arbitrary finite length when $g_i \in G$, $h_i \in H$. 
where \( g_i \in G - \mathcal{E}_{13} \) for each \( i \) and \\
\( h_i \in H - \mathcal{E}_{13} \) for each \( i \).

under the operation of juxtaposition

Given \((a_i, b_i, a_2 b_2 \ldots a_m b_m) \ast (c_i, d_i, \ldots c_n, d_n) = (a_i, b_i \ldots a_m b_m, c_i, d_i, \ldots c_n, d_n)\)

Group axioms

Identity : \( G \ast H \) is defined to contain the empty word, \( \emptyset \)

Closure : Two words juxtaposed always give a word and after we reduce the word we get an element of \( G \ast H \)

Ex. \( a \in G \) and \( b \in H \)

\[ b^{-1}a^{-1}, a b \in G \ast H \]

\[ a b b^{-1}a^{-1} \notin G \ast H \]

\[ a a^{-1} \notin G \ast H \]

\[ \emptyset \in G \ast H \]

Inverse element :

\[ a_i, b_i, a_2 b_2 \ldots a_m b_m \ast b_m^{-1}a_n^{-1} \ldots b_1^{-1}a_1^{-1} = \emptyset \]

Associativity : Too hard for us!
Examples

$\mathbb{Z} \ast \mathbb{Z} \equiv \langle a \rangle \ast \langle b \rangle$

$aba^2b^3a^{-1}ba^{-2} \in \mathbb{Z} \ast \mathbb{Z}$

This is the free group on two generators.

$\mathbb{Z}_2 \ast \mathbb{Z}_2 \equiv \langle a \mid a^2 = 1 \rangle \ast \langle b \mid b^2 = 1 \rangle$

$ababa, \; ab \in \mathbb{Z}_2 \ast \mathbb{Z}_2$

Definition: The free group on $n$ generators is defined to be $\mathbb{Z} \ast \mathbb{Z} \ast \ldots \ast \mathbb{Z}$, $n$-times.

Universal property of free products

Given groups $G_1, G_2$ and $H$ and homomorphisms $\varphi_1: G_1 \to H$ and $\varphi_2: G_2 \to H$, there exists a unique homomorphism $\varphi: G_1 \ast G_2 \to H$ s.t.

for every one letter word $g \in G_1 \ast G_2$ s.t. $g \in G_1$ $\varphi(g) = \varphi_1(g)$

for every one letter word $g \in G_2$ $\varphi(g) = \varphi_2(g)$.

Namely, $\varphi(a_1b_1a_2b_2 \cdots a_m b_m) = \varphi_1(a_1) \varphi_2(b_1) \varphi_1(a_2) \cdots \varphi_2(b_m)$.

In particular, the inclusions $G \hookrightarrow G \times H$ and $H \hookrightarrow G \times H$ induce a unique (surjective) homomorphism $G \ast H \to G \times H$. 
Def 1. Given a group \((G, \ast)\) a subset of \(G\) denoted by \(S \subseteq G\) is a generating set for \(G\) if every element in \(G\) can be expressed as a combination of elements in \(S\) under the operation \(\ast\) and their inverses.

Ex 1. What is a generating set for:
\[\mathbb{Z} \oplus \mathbb{Z} \mathbb{Z} \mathbb{Z} \mathbb{Z} \{0, 1\}, \{1, 0\} \mathbb{Z} \mathbb{Z} \mathbb{Z} \mathbb{Z}\]

What is a generating set for \(S_4\) (the symmetric group on 4 letters)
\[\{1 2\}, \{2 3\}, \{3 4\}\]

Def 1. A group presentation \(<S | R>\)

is a set of generator \(S\) and a set of reduced words in \(S\) and the inverses of elements in \(S\).

As a group \(<S | R> \cong \frac{F_S}{\langle \langle R \rangle \rangle}\)

where \(F_S\) is the free group on \(S\) and \(\langle \langle R \rangle \rangle\) is the normal subgroup of \(F_S\) generated by elements of \(R\).
Examples

$\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \langle a | a^2 \rangle$

$\mathbb{Z}_n = \langle a | a^n \rangle$

$\mathbb{Z} = \langle a | \rangle$

$\mathbb{F}_n = \langle a | \rangle$

$\mathbb{Z} \oplus \mathbb{Z}_n = \langle a, b | b^n, aba^{-1} b^{-1} \rangle$

$\mathbb{Z} \ast \mathbb{Z} = \langle a, b | \rangle$

$S_4 = \langle \sigma_1, \sigma_2, \sigma_5 | \sigma_1^2, \sigma_2^2, \sigma_5^2, \sigma_1 \sigma_3 \sigma_1^{-1} \sigma_5^{-1},$

$\sigma_1 \sigma_2 \sigma_1 = \sigma_2, \sigma_1, \sigma_2,$

$\sigma_2 \sigma_5 \sigma_2 = \sigma_5 \sigma_2 \sigma_3 \rangle$

$B_4 = \text{braid group on 4 strands} = \langle \sigma_1, \sigma_2, \sigma_5 | \sigma_1 \sigma_5 \sigma_1^{-1} \sigma_5^{-1},$

$\sigma_1 \sigma_2 \sigma_1 = \sigma_2, \sigma_1, \sigma_2,$

$\sigma_2 \sigma_5 \sigma_2 = \sigma_5 \sigma_2 \sigma_3 \rangle$

$\mathbb{Z}_2 \ast \mathbb{Z}_2 = \langle a, b | a^2 = 1, b^2 = 1 \rangle \cong \frac{\mathbb{F}_2 \ast \mathbb{F}_2}{\langle a^2, b^2 \rangle}$