Announcements

- HW 5 posted today - HW 4 Due today.
- Midterm scores

50 − 35 A	34 − 25 B	24 − 15 C	Ave ≅ 29/50
2	8	6	SD ≅ 9

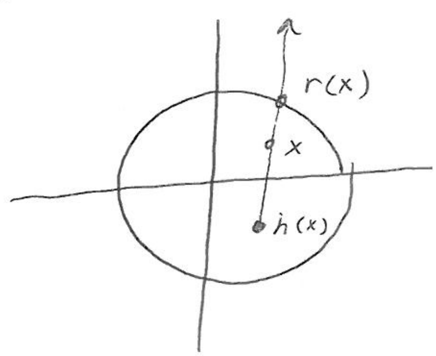
Outline

- Review
- Fundamental Theorem of Algebra

Last time

Th^m (H:1.9) Every continuous map $h: D^2 \to D^2$ has a fixed point.

Proof outline: If not, then we have a retraction from D^2 to S^1. $r: D^2 \to S^1$.



Fundamental Th^m of Algebra

Th^m (H: 1.8) Every non-constant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

Pf] Suppose to form a contradiction that

$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ has no roots in $\mathbb{C}$.

For each real number r, define

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$$

Since $p(z) \neq 0$, $f_r : I \to S^1$ is continuous for each $r \in \mathbb{R}$.

Moreover, since $p(z) \neq 0$, $\cancel{f_r(s)}$ $f(r,s) : \mathbb{R} \times I \to S^1$ for $f(r,s) = f_r(s)$ is continuous.

$$f_r(0) = \frac{p(r)/p(r)}{|p(r)/p(r)|} = 1 \qquad f_r(1) = \frac{p(r)/p(r)}{|p(r)/p(r)|} = 1.$$

Hence, for fixed r, $f_r(s)$ is a path in S^1 based at 1.

$$f_0(s) = \frac{p(0)/p(0)}{|p(0)/p(0)|} = 1$$ Hence $f_0(s)$ is the constant map.

Fix $R \in \mathbb{R}$.

Define a path homotopy $H : I \times I \to S^1$ via

$$H(s,t) = f_{Rt}(s)$$

from $f_0(s)$ to $f_R(s)$. Note that R is arbitrary.

So $f_R(s)$ is path homotopic to the constant map for any $R \in \mathbb{R}$.

Fix R s.t. $R > \max\{|a_1| + \cdots + |a_n|, 1\}$

Choose $z \in \mathbb{C}$ s.t. $|z| = R$.

$$|z^n| = R^n = R \cdot R^{n-1} > (|a_1| + \cdots + |a_n|)|z^{n-1}|$$

triangle inequality $\cancel{\longrightarrow |a_1 z^{n-1} + a_2 z|}$

$$\geq |a_1||z^{n-1}| + |a_2||z^{n-1}| + \cdots + |a_n||z^{n-1}|$$

Since $R > 1$

$$> |a_1 z^{n-1}| + |a_2 z^{n-2}| + \cdots + |a_n|$$

by triangle inequ. $\rightarrow$ $> |a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n|$

Since $|z^n| > |a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n|$ for all $z \in \mathbb{C}$ s.t. $|z| = R$

Then $p_t(z) = z^n + t(a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n)$ has no roots for all z s.t. $|z| = R$ and for all t s.t. $t \in [0,1]$.

Hence $g_t(s) = \dfrac{p_t(Re^{2\pi i s})/p_t(R)}{|p_t(Re^{2\pi i s})/p_t(R)|}$ is continuous ~~in~~

$$g: \cancel{S_R \times I}\ I \times I \to S^1$$

$$g_0(s) = \frac{R^n e^{2\pi i n s}/R^n}{|R^n e^{2\pi i n s}/R^n|} = e^{2\pi i n s}$$

$$g_1(s) = f_R(s)$$

Hence $g_0(s) \simeq_p f_R(s) \simeq_p e_1$ $\leftarrow$ constant map.

Announcements | *Record*

- HW 5 due a week from today
- Colloquium at noon on Friday F03-200A
  "Width of a 3-manifold"

Outline |

- Grad school talk
- Munkres 58 Deformation retracts and Homotopy type.

Recall | ~~A ret~~ Given $A \subset X$ a retration of X onto A is a continuous map $h: X \to A$ s.t. $h|_A = id_A$.

Lemma 58.1 | Let $h, k: (X, x_0) \to (Y, y_0)$ be continuous maps. If h is homotopic to k via a homotopy that always sends x_0 to y_0, then $h_* = k_*$.

Pf | There exists a homotopy $H: X \times I \to Y$ from h to k s.t. $H(x_0, t) = y_0$ for all $t \in I$.

Let $f: I \to X$ be a loop based at x_0.

Claim $H \circ (f \times id_I): I \times I \to Y$ is a path homotopy

1. Continuous since compositions and products of continuous functions are continuous.
2. $H \circ (f \times id_I)(s, 0) = H(f(s), 0) = h \circ f(s)$
3. $H \circ (f \times id_I)(s, 1) = H(f(s), 1) = k \circ f(s)$
4. $H \circ (f \times id_I)(0, t) = H(x_0, t) = y_0$
5. $H \circ (f \times id_I)(1, t) = H(x_0, t) = y_0$

Hence $h \circ f \simeq_p k \circ f$.

$$[h \circ f] = [k \circ f] \text{ as elements in } \pi_1(Y, y_0)$$

$$h_*([f]) = k_*([f])$$

Since f was an arbitrary loop based at x_0, $h_* = k_*$ $\square$

Thm | (58.2) The inclusion $i: S^1 \to \mathbb{R}^2 - \{0\}$ induces an isomorphism on fundamental groups.

Pf | Recall, we have showed that $r: \mathbb{R}^2 - \{0\} \to S^1$ given by $r(\vec{x}) = \dfrac{\vec{x}}{\|\vec{x}\|}$ is a retraction. Hence, by Lemma (55.1), i_* is one-to-one.

Examine the map $i \circ r: \mathbb{R}^2 - \{0\} \to \mathbb{R}^2 - \{0\}$

$$\mathbb{R}^2 - \{0\} \xrightarrow[r]{} S^1 \xrightarrow[i]{} \mathbb{R}^2 - \{0\}.$$

Claim: $i \circ r$ is homotopic to $id_{\mathbb{R}^2 - \{0\}}$

Pf | $H: \mathbb{R}^2 - \{0\} \times I \to \mathbb{R}^2 - \{0\}$

$$H(\vec{x}, t) = (1-t)\frac{\vec{x}}{\|\vec{x}\|} + t\vec{x}$$

$*$ Check this is a homotopy.

Note that $H(\langle 1,0\rangle, t) = (1-t)\overset{\langle 1,0\rangle}{} + t\langle 1,0\rangle = \langle 1,0\rangle$

Thus, by Lemma 58.1

$$(i \circ r)_* = \left(id_{\mathbb{R}^2 - \{0\}}\right)_*$$

$$i_* \circ r_* = id_{\pi_1(\mathbb{R}^2 - \{0\}, \langle 1,0\rangle)}$$

Hence i_* must be onto.

Since i_* is a bijective homomorphism i_* is an isomorphism

<u>Def</u> Let $A \subset X$. A is a <u>deformation retract</u> of X if there is a continuous map $H: X \times I \to X$ s.t. $H(x,0) = x$, $H(x,1) \in A$ for all $x \in X$ and $H(a,t) = a$ for all $a \in A$. The homotopy H is called a <u>deformation retraction</u> of X onto A.

Note $H: \mathbb{R}^2 - \{\vec{0}\} \times I \to \mathbb{R}^2 - \{\vec{0}\}$ given by

$$H(\vec{x}, t) = (1-t)\vec{x} + t\frac{\vec{x}}{\|\vec{x}\|}$$ is a deformation retract.

① $H(\vec{x},0) = \cancel{\vec{x}}\ (1-0)\vec{x} + 0\frac{\vec{x}}{\|\vec{x}\|} = \vec{x}$

② $H(\vec{x},1) = (1-1)\vec{x} + 1 \cdot \frac{\vec{x}}{\|\vec{x}\|} = \frac{\vec{x}}{\|\vec{x}\|} \in S^1$

③ $$\begin{aligned} H(\vec{a},t) &= (1-t)\vec{a} + t\frac{\vec{a}}{\|\vec{a}\|} \\ &= (1-t)\vec{a} + t\vec{a} \\ &= \vec{a} \end{aligned}$$

<u>Thm</u> (58.3) Let A be a deformation retract of X. Let $x_0 \in A$. The inclusion map $j: (A, x_0) \to (X, x_0)$ induces an isomorphism on fundamental groups.

<u>Pf</u> Follows from the proof of Lemma 58.2
(This is a good exercise)

Def Let $f: X \to Y$ and $g: \cancel{X} \to X$ be continuous maps. If $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$ then f and g are <u>homotopy equivalences</u>, X is <u>homotopic</u> to Y and f is the homotopy inverse of g.

Exercise: Show homotopy equivalence is an equivalence relation on topological spaces.

Hence, the <u>homotopy type</u> of X is the equivalence class of all top. spaces homotopic to X.

Building toward a big theorem

Thm (58.7) If X is homotopic to Y where X and Y are path-connected, then

$$\pi_1(X, x_0) \cong \pi_1(Y, y_0).$$