Announcements
- HW 5 posted today - HW 4 Due today.
- Midterm scores:
  50 - 35 A
  34 - 25 B
  24 - 15 C
  14 - 0 D
  Average 29/50
  SD ≈ 9

Outline
- Review
- Fundamental Theorem of Algebra

Last time

Theorem (H:1.9) Every continuous map \( h: D^2 \to D^2 \) has a fixed point.

Proof outline: If not, then we have a retraction from \( D^2 \) to \( S^1 \).

\[
\begin{align*}
n(x) & \quad r(x) \\
n(y) & \quad r(y)
\end{align*}
\]

Fundamental Theorem of Algebra

Theorem (H:1.8) Every non-constant polynomial with coefficients in \( \mathbb{C} \) has a root in \( \mathbb{C} \).
Suppose to form a contradiction that $p(z) = z^n + a_1z^{n-1} + \cdots + a_n$ has no roots in $\mathbb{C}$.

For each real number $r$, define

$$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|}$$

Since $p(z) \neq 0$, $f_r : I \to S'$ is continuous for each $r \in \mathbb{R}$.

Moreover, since $p(z) \neq 0$, $f(r, s) : \mathbb{R} \times I \to S'$ for $f(r, s) = f_r(s)$ is continuous.

$$f_r(0) = \frac{p(r)/p(r)}{|p(r)/p(r)|} = 1 \quad f_r(1) = \frac{p(r)/p(r)}{|p(r)/p(r)|} = 1.$$

Hence, for fixed $r$, $f_r(s)$ is a path in $S'$ based at $1$.

$$f_0(s) = \frac{p(0)/p(0)}{|p(0)/p(0)|} = 1 \quad \text{Hence } f_0(s) \text{ is the constant map.}$$

Fix $R \in \mathbb{R}$.

Define a path homotopy $H : I \times I \to S'$ via

$$H(s, t) = f_{Rt}(s)$$

from $f_0(s)$ to $f_R(s)$. Note that $R$ is arbitrary.

So $f_R(s)$ is path homotopic to the constant map for any $R \in \mathbb{R}$.
Fix $R \geq 1$. Let $R > \max \{ |a_1|, \ldots, |a_n|, 1 \}$.

Choose $z \in \mathbb{C}$ s.t. $|z| = R$. Then

$$|z^n| = R^n = R \cdot R^{n-1} > (|a_1| + \cdots + |a_n|) |z^{n-1}|$$

by the triangle inequality.

Then

$$|a_1 z^{n-1}| + |a_2 z^{n-2}| + \cdots + |a_n|$$

Since $R > 1$,

$$|a_1 z^{n-1}| + |a_2 z^{n-2}| + \cdots + |a_n|$$

by the triangle inequality.

Since $|z^n| > |a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n|$, for all $z \in \mathbb{C}$ s.t. $|z| = R$ and for all $t \in [0, 1]$.

Then $p_t(z) = z^n + t(a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n)$ has no roots.

Hence

$$g_t(s) = \frac{p_t(R e^{2\pi i s})}{p_t(R)}$$

is continuous.

$$g_0(s) = \frac{R^n}{R^n e^{2\pi i s}} = e^{2\pi i s}$$

$$g_1(s) = f_R(s)$$

Hence

$$g_0(s) \preceq_R f_R(s) \preceq_R e_1$$

constant map.
Announcements

Record

- HW 5 due a week from today
- Colloquium at noon on Friday F03-200A
  "Width of a 3-manifold"

Outline

- Grad school talk
- Munkres 58 Deformation retracts and Homotopy type.

Recall. Given $A \subset X$ a retraction of $X$ onto $A$ is a continuous map $h : X \to A$ s.t. $h|_A = \text{id}_A$.

Lemma 58.1. Let $h, k : (x, x_0) \to (y, y_0)$ be continuous maps. If $h$ is homotopic to $k$ via a homotopy that always sends $x_0$ to $y_0$, then $h_* = k_*$.

Proof. There exists a homotopy $H : X \times I \to Y$ from $h$ to $k$ s.t. $H(x_0, t) = y_0$ for all $t \in I$.

Let $f : I \to X$ be a loop based at $x_0$.

Claim: $H_0(f \times id_I) : I \times I \to Y$ is a path homotopy

1. Continuous since compositions and products of continuous functions are continuous.
2. $H_0(f \times id_I)(s, 0) = H(f(s), 0) = h(f(s))$
3. $H_0(f \times id_I)(s, 1) = H(f(s), 1) = k(f(s))$
4. $H_0(f \times id_I)(0, t) = H(x_0, t) = y_0$
5. $H_0(f \times id_I)(1, t) = H(x_0, t) = y_0$
Hence \( h_{\text{of}} \succ \! \succ \kappa_{\text{of}} \).

\[
[h_{\text{of}}] = [k_{\text{of}}] \text{ as elements in } \pi_{1}(Y, y_0)
\]

\[
h_{\ast}([I_{f}]) = k_{\ast}([I_{f}])
\]

Since \( f \) was an arbitrary loop based at \( x_0 \), \( h_{\ast} = k_{\ast} \) \( \square \)

\[\text{Theorem (58.2)}\]
The inclusion \( i: S^1 \to \mathbb{R}^2 - \{0\} \) induces an isomorphism on fundamental groups.

\[\text{Proof}\]
Recall, we have showed that \( r: \mathbb{R}^2 - \{0\} \to S^1 \)
given by \( r(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|} \) is a retraction. Hence, by Lemma (55.01), \( i_{\ast} \) is one-to-one.

Examine the map \( i \circ r: \mathbb{R}^2 - \{0\} \to \mathbb{R}^2 - \{0\} \)

\[
\mathbb{R}^2 - \{0\} \xrightarrow{r} S^1 \xrightarrow{i} \mathbb{R}^2 - \{0\}.
\]

Claim: \( i \circ r \) is homotopic to \( \text{id}_{\mathbb{R}^2 - \{0\}} \)

\[\text{Proof}\]
\( H: \mathbb{R}^2 - \{0\} \times I \to \mathbb{R}^2 - \{0\} \)

\[
H(x, t) = (1-t) \frac{x}{\|x\|} + t \hat{x}
\]

\(*\) Check this is a homotopy.

Note that \( H(<1,0>, t) = (1-t) <1,0> + t <1,0> = <1,0> \)

Thus, by Lemma 58.1,

\[
(i \circ r)_{\ast} = (\text{id}_{\mathbb{R}^2 - \{0\}})_{\ast}
\]

\[i_{\ast} \circ r_{\ast} = \text{id}_{\pi_{1}(\mathbb{R}^2 - \{0\}, <1,0>)}\]
Hence \( i^* \) must be onto.
Since \( i^* \) is a bijective homomorphism, it is an isomorphism.

**Def.** Let \( A \subset X \). \( A \) is a deformation retract of \( X \) if there is a continuous map \( H : X \times I \to X \) s.t.
\[
H(x, 0) = x, \quad H(x, 1) \in A \text{ for all } x \in X \text{ and } H(a, t) = a \text{ for all } a \in A.
\]
The homotopy \( H \) is called a deformation retraction of \( X \) onto \( A \).

Note \( H : \mathbb{R}^2 \times \mathbb{R} \times I \to \mathbb{R}^2 \times \mathbb{R} \) given by
\[
H(x, t) = (1-t)x + t \frac{x}{\|x\|}
\]
is a deformation retract.

1. \( H(x, 0) = x = (1-0)x + 0 \frac{x}{\|x\|} = x \)
2. \( H(x, 1) = (1-1)x + 1 \cdot \frac{x}{\|x\|} = \frac{x}{\|x\|} \in S^1 \)
3. \( H(a, t) = (1-t)a + t \frac{a}{\|a\|} = (1-t)a + t \frac{a}{\|a\|} = a \)

(58.0.3)

**Thm.** Let \( A \) be a deformation retract of \( X \).
Let \( x_0 \in A \). The inclusion map \( j : (A, x_0) \to (X, x_0) \) induces an isomorphism on fundamental groups.

**Pf.** Follows from the proof of Lemma 58.0.2
(This is a good exercise)
Def: Let $f: X \to Y$ and $g: Y \to X$ be continuous maps. If $gof = id_X$ and $fog = id_Y$, then $f$ and $g$ are homotopy equivalences, $X$ is homotopic to $Y$, and $f$ is the homotopy inverse of $g$.

[Exercise: Show homotopy equivalence is an equivalence relation on topological spaces.]

Hence, the homotopy type of $X$ is the equivalence class of all top spaces homotopic to $X$.

Building toward a big theorem

Thm (58.7) If $X$ is homotopic to $Y$, where $X$ and $Y$ are path-connected, then $\pi_1 (X, x_0) \cong \pi_1 (Y, y_0)$. 