550 B Lec. 1

Outline

- Syllabus
- Review 550A concepts
    - Connected
    - Path-connected
    - Quotient space
    - Manifold

Connectedness

Let X be a top. space.

Def | A separation of X is a pair of non-empty, disjiont, open sets U and V s.t. $X = U \cup V$. We write it as $X = U | V$. If X has no separation we say X is connected.

Examples | • $\mathbb{R}$ is connected. ← Comp question.

• Any set with at least two points and the discrete topology has a separation.

Big Theorems

Thm | If $X = A | B$ and Y is a connected subspace of X, then $Y \subset A$ or $Y \subset B$.

Thm | If X and Y are connected, then $X \times Y$ is connected.

Thm | The image of a connected space under a continuous map is connected.

Proof | Let X be a connected top. space and let $f: X \to Y$ be a continuous map. Let $Z = f(X) \subset Y$ and define the ~~restriction of f to~~ $f^*: X \to Z$.

By basic properties of continuous maps f^* is continuous. Suppose, to form a contradiction, $Z = A | B$.

Claim: $X = (f^*)^{-1}(A) | (f^*)^{-1}(B)$.

Since $Z = A \cup B$, then $X = (f^*)^{-1}(A \cup B) = (f^*)^{-1}(A) \cup (f^*)^{-1}(B)$

Since $A \cap B = \emptyset$ and f^* is a function, then $(f^*)^{-1}(A) \cap (f^*)^{-1}(B) = \emptyset$.

Since f^* is continuous, both $(f^*)^{-1}(A)$ and $(f^*)^{-1}(B)$ are open.

Since $A \neq \emptyset$ and $B \neq \emptyset$ and f^* is <u>onto</u>, then $(f^*)^{-1}(A) \neq \emptyset$ and $(f^*)^{-1}(B) \neq \emptyset$.

Hence $(f^*)^{-1}(A)$ and $(f^*)^{-1}(B)$ are non-empty, disjiont, open sets s.t. $X = (f^*)^{-1}(A) \cup (f^*)^{-1}(B)$. This is a contradiction to the fact that X is connected. Hence $Z = f(X)$ is connected. $\square$

Path-connectedness.

Let X be a top. space.

Given $x, y \in X$, a *path* in X from x to y is a continuous map $f: [0,1] \to X$ s.t. $f(0) = x$ and $f(1) = y$.

X is *path-connected* if for every pair of points $x, y \in X$ there exists a path in X from x to y.

Big theorems

Thm/ If X is path-connected, then X is connected.

Thm/ If X and Y are path-connected, then $X \times Y$ is path-connected.

Thm/ Not every connected space is path-connected Comp. question

(i.e. topologists sign curve).

Proof/ Suppose X is path connected and $X = A | B$, in search of a contradiction. Since $A \neq \emptyset$ and $B \neq \emptyset$, let $a \in A$ and $b \in B$. Since X is path connected $\exists$ a continuous function $f: I \to X$ s.t. $f(0) = a$ and $f(1) = b$. Recall, I is connected. Since the continuous image of connected sets is connected $f(I)$ is connected. Since $f(I)$ is connected $f(I) \subset A$ or $f(I) \subset B$. If $f(I) \subset B$, then we contradict $f(0) \in A$ and $A \cap B = \emptyset$. If $f(I) \subset A$, then we contradict $f(1) \in B$ and $A \cap B = \emptyset$. $\square$

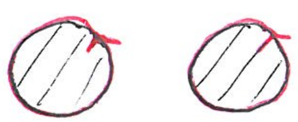
Quotient Spaces.

Def Let X and Y be top. spaces. Let $p: X \to Y$ be a surjective map. p is a *quotient map* if $V \subset Y$ is open *iff* $p^{-1}(V)$ is open in X.

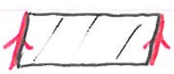
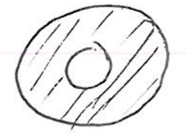
Given a surjective map $p: X \to Y$ with X a top space and Y a set. $\exists!$ topology on Y s.t. p is a quotient map. Call this topology the quotient topology for p.

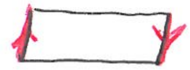
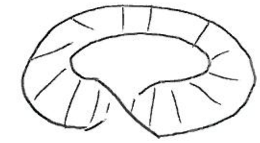
Let $\sim$ be an equivalence relation on X and let X^* be the *set* of equivalence classes. $p: X \to X^*$ given by $p(x) = [x]_\sim$ is a surjective map. Hence $\exists!$ topology on X^* that makes p a quotient map.

This gives us all of the formalism to make rigorous the notion of gluing.

 $= D_1^2 \amalg D_2^2 / \sim \;\cong S^2$

$e^{i\theta}$ in ∂D_1^2 is equivalent to $e^{i\varphi} \in \partial D_2^2$ if $\theta = \varphi$

 $\cong S^1 \times I =$ annulus $=$ 

 $\cong$ Mobius band $= S^1 \tilde{\times} I =$ 

$$D_1^2 \times S_1^1 \amalg D_2^2 \times S_2^1 \big/ \sim$$

Make the identifications $\partial D_1^2 \ni e^{i\alpha}$ $\quad 0 \leq \alpha < 2\pi$

$$\partial D_2^2 \ni e^{i\beta}$$

$$S_1^1 \ni e^{i\gamma}$$

$$S_2^1 \ni e^{i\delta}$$

$$\partial(D_1^2 \times S_1^1) = \partial D_1^2 \times S_1^1 \ni (e^{i\alpha}, e^{i\gamma})$$

Similarly $\partial(D_2^2 \times S_2^1) \ni (e^{i\beta}, e^{i\delta})$

$$(e^{i\alpha}, e^{i\gamma}) \sim (e^{i\beta}, e^{i\delta}) \text{ if } \beta = \alpha \text{ and } \gamma = \delta.$$

- This specifies the way of gluing the boundary of these objects together. In this case, the quotient is homeomorphic to $S^2 \times S^1$

- If instead $(e^{i\alpha}, e^{i\gamma}) \sim (e^{i\beta}, e^{i\delta})$ if $\alpha = \delta$ and $\beta = \gamma$

then the quotient is homeomorphic to S^3.

Homotopy

Let X, Y be top. spaces and $f, f': X \to Y$ be two continuous maps. We say f is <u>homotopic</u> to f' if there is a continuous map $F: X \times I \to Y$ s.t.

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = f'(x).$$

- F is called a <u>homotopy</u> between f and f'.
- If f is homotopic to f', we write $f \simeq f'$.
- If f is homotopic to the constant map, we say f is nulhomotopic.

You can think of a homotopy as a continuous deformation of one map to another.

Recall

A <u>path</u> in X from x_0 to x_1 is a continuous map $f: [0,1] \to X$ s.t. $f(0) = x_0$ and $f(1) = x_1$.

<u>Def</u> Two paths f and f' in X are <u>path homotopic</u> if there is a homotopy $F: I \times I \to X$ between f and f' s.t.

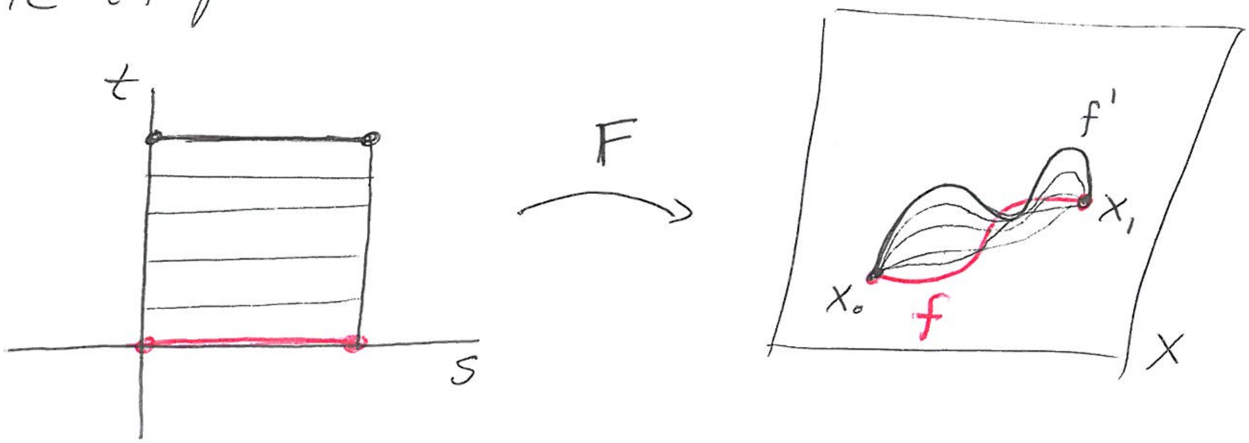
$$F(0,t) = x_0 = f(0) = f'(0) \text{ for all } t \in I$$

and

$$F(1,t) = x_1 = f(1) = f'(1) \text{ for all } t \in I.$$

If f is path-homotopic to f' we write $f \simeq_p f'$.

Pic of path homotopy



Lemma 51.1 The relations $\simeq$ and $\simeq_p$ are equivalence relations.

Pf Show $\simeq$ is an equivalence relation.

1. Let $f: X \to Y$ be continuous. $f \simeq f$ via the homotopy $F: X \times I \to Y$ via $F(x,t) = f(x)$.
2. Suppose $f \simeq f': X \to Y$ via the homotopy $F(x,t)$. Then $F(x, 1-t)$ is a homotopy between f' and f. Hence $f' \simeq f$.
3. Suppose $f \simeq f'$ via homotopy $F(x,t)$ and $f' \simeq f''$ via homotopy $F'(x,t)$. Define $G: X \times I \to Y$ by

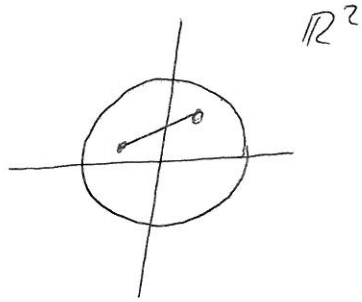
$$G(x,t) = \begin{cases} F(x, 2t) & 0 \le t \le 1/2 \\ F'(x, 2t-1) & 1/2 \le t \le 1 \end{cases}$$

Note G is well-defined as $F(x,1) = f'(x) = F'(x,0)$ for all x.

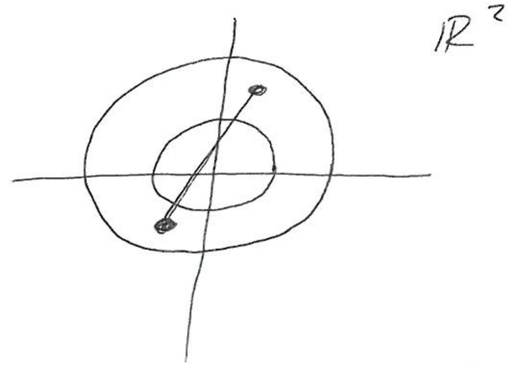
By the pasting lemma, since G is continuous on closed sets $X \times [0, 1/2]$ and $X \times [1/2, 1]$, then it is continuous on all of $X \times I$. Hence G is a homotopy between f and f''. □

Exercize: Prove the lemma for $\simeq_p$.

Def A set $A \subset \mathbb{R}^n$ is convex if for every pair of distinct points $a, b \in A$, the straight line segment $[a,b]$ connecting them is entirely contained in A.



disk is convex



The annulus is <u>not</u> convex

Prop Let X be a top. space and let $A \subset \mathbb{R}^n$ be a convex subset. Let $f: X \to A$ and $g: X \to A$ be continuous maps. Then $f \simeq g$.

Pf ~~the str~~ Claim: The <u>straight line homotopy</u>

$$F: X \times I \to A \quad \text{s.t. } F(x,t) = (1-t)f(x) + tg(x)$$

is a homotopy between f and g.

① F is well-defined since A is convex.

② F is continuous since addition and multiplication of continuous functions with domain in $\mathbb{R}^m$ are continuous.

③ $F(x,0) = (1)f(x) + 0g(x) = f(x)$

$F(x,1) = (1-1)f(x) + 1g(x) = g(x)$

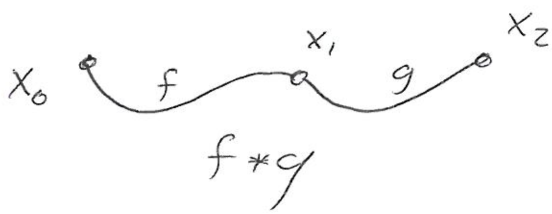
Hence $f \simeq g$.

Product of paths

Let X be a top. space. Let f be a path in X between x_0 and x_1 and let g be a path between x_1 and x_2. Define

$$f*g(s) = h(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1 \end{cases}$$

Note $f*g$ is well defined since $f(2(1/2)) = x_1 = g(2(1/2)-1)$, and $f*g$ is continuous by the pasting lemma. Hence $f*g$ is a path from x_0 to x_2.



Let $[f]$ and $[g]$ be the path-homotopy classes of f and g in X. Define $[f]*[g] = [f*g]$.

Prop| The operation $[f]*[g]$ is well defined.

Pf| Let $f, f' \in [f]$
Let $F: I \times I \to X$ be the path homotopy between f and f'
Let $g, g' \in [g]$.
Let $G: I \times I \to X$ be the path homotopy between g and g'.

Claim: $H(s,t) = \begin{cases} F(2s, t) & \text{for } s \in [0, 1/2] \\ G(2s-1, t) & \text{for } s \in [1/2, 1] \end{cases}$

Is a path homotopy between $f*g$ and $f'*g'$

Pf| Exercise.