1. Munkres §54 Exercise 3

Let \( p : E \to B \) be a covering map, and let \( \alpha \) and \( \beta \) be paths in \( B \), such that \( \alpha(1) = \beta(0) \). Let \( \hat{\alpha} \) and \( \hat{\beta} \) be liftings of \( \alpha \) and \( \beta \) respectively such that \( \hat{\alpha}(1) = \hat{\beta}(0) \). Prove that \( \hat{\alpha} \ast \hat{\beta} \) is a lifting of \( \alpha \ast \beta \).

**Proof.** Since \( \hat{\alpha} \ast \hat{\beta} \) is a continuous map from \([0, 1]\) to \( E \) by construction (per discussion on page 325), what we must show is that

\[
p \circ (\hat{\alpha} \ast \hat{\beta}) = \alpha \ast \beta.
\]

If \( x \in [0, \frac{1}{2}] \), then

\[
(\alpha \ast \beta(x))
\]

while if \( x \in [\frac{1}{2}, 1] \), then

\[
(p \circ (\hat{\alpha} \ast \hat{\beta}))(x) = p(\hat{\beta}(x)) = \beta(x) = \alpha \ast \beta(x),
\]

so that in general \( x \in [0, 1] \) \( \implies (p \circ (\hat{\alpha} \ast \hat{\beta}))(x) = (\alpha \ast \beta)(x) \). Since \( x \) was arbitrary in \([0, 1]\), we obtain

\[
p \circ (\hat{\alpha} \ast \hat{\beta}) = \alpha \ast \beta.
\]

It follows that \( \hat{\alpha} \ast \hat{\beta} \) is a lift of \( \alpha \ast \beta \).

Note: The requirement that \( \hat{\alpha}(1) = \hat{\beta}(0) \) was needed for \( \hat{\alpha} \ast \hat{\beta} \) to be well-defined; however, if \( \hat{\alpha}(1) \neq \hat{\beta}(0) \), the piecewise function

\[
f(x) = \begin{cases} 
\hat{\alpha}(x) & x \in [0, \frac{1}{2}] \\
\hat{\beta}(x) & x \in (\frac{1}{2}, 1]
\end{cases}
\]

is still a lift of \( \alpha \ast \beta \), albeit not a path in the covering space \( E \).
2. Munkres §54 Exercise 6

Proof. Let \( G \) denote \( \pi(S^1, b_0) \). Since \( G \cong \pi(S^1, a_0) \cong \mathbb{Z} \), for all \( a_0 \) in \( S^1 \) (by the path connectedness of \( S^1 \)), we will take \( b_0 = (1, 0) \) whenever explicit calculations occur. What is necessary and sufficient for computing the homomorphisms \( g^* \) and \( h^* \) is to determine what they do to a generator of \( \pi(S^1, b_0) \). (It is an elementary fact from algebra that a homomorphism is uniquely determined by its action on generators, and \( \mathbb{Z} \) is generated by a single element, 1.) A generator for \( G \) is furnished by \( \Phi^{-1}(1) \), where \( \Phi \) is the isomorphism of Theorem 54.5. Per the definition on page 345 (Theorems 54.4 and 5), \( \Phi^{-1}(1) \) is the equivalence class of loops based at \( (1,0) \) such that for any representative \( f \) of the class, the lifting \( \tilde{f} \) that begins at 0, with respect to the covering map \( p \) given in Theorem 53.1, has endpoint 1. Define the path \( f \) in \( S^1 \) by \( f(x) = e^{2\pi i x} \). Note that \( p[\alpha](x) = f(x) \), so that \( p \circ \alpha = f \), where \( \alpha \) is the identity map on \([0, 1]\). It follows that \( \alpha \) is the unique lifting of \( f \) that begins at 0. Therefore, \( \tilde{f} = \alpha \), and \( \tilde{f}(1) = \alpha(1) = 1 \). Thus, \( \Phi([f]) = 1 \) and \( \Phi^{-1}(1) = [f] \). So \([f]\) generates \( G \), and \( g^* \) is determined by \( g^*([f]) = [g \circ f] \).

We compute
\[
g \circ f(x) = (e^{2\pi i x})^n = e^{2\pi inx}.
\]

We will now lift this map to \( \mathbb{R} \) via \( p \).

Let \( k : [0,1] \to \mathbb{R} \) by \( k(x) = nx \). Then for all \( x \in [0,1] \),
\[
p \circ k(x) = e^{2\pi inx} = (g \circ f)(x),
\]
so that \( k \) is a lifting of \( g \circ f \), and hence the unique such lifting beginning at 0. In other words, \( k = g \circ f \). Further \( k(1) = n \). Thus,
\[
\Phi(g^*([f])) = \Phi([g \circ f]) = k(1) = n = n \cdot 1.
\]

We can finally conclude that
\[
g^*([f]) = \Phi^{-1} \circ \Phi(g^*([f]))
= \Phi^{-1}(n \cdot 1)
= n \Phi^{-1}(1)
= n \cdot [f]
= [f] \ast [f] \ast \ldots \ast [f] \ (n \text{ times}).
\]

Or, if we identify \( G \) with \( \mathbb{Z} \), the map \( g^* \) is the (unique) homomorphism from \( \mathbb{Z} \) to \( \mathbb{Z}/n\mathbb{Z} \) given by \( 1 \mapsto n \).

We now compute \( h^* \) while keeping in mind that \([f]\) generates \( G \).
\[
h^*([f]) = [h \circ f], \quad \text{and}
\]
\[
h \circ f(x) = (e^{2\pi i x})^{-n} = e^{-2\pi inx}.
\]
Let \( l : [0, 1] \to \mathbb{R} \) by \( l(x) = -nx \). Then for all \( x \in [0, 1] \),
\[
p \circ l(x) = e^{-2\pi i nx} \cdot (h \circ f)(x),
\]
so that \( l \) is a lifting of \( h \circ f \), and hence the unique such lifting beginning at 0. In other words, \( l = \overline{h} \circ \overline{f} \). Further \( l(1) = -n \). Thus,
\[
\Phi(h^*(f)) = \Phi([h \circ f]) = l(1) = -n = n \cdot -1.
\]
We conclude that
\[
h^*(f) = \Phi^{-1} \circ \Phi(h^*(f))
\]
\[= \Phi^{-1}(n \cdot -1)
\]
\[= n\Phi^{-1}(-1)
\]
\[= n \cdot [\overline{f}]
\]
\[= [\overline{f}] * [\overline{f}] * \cdots * [\overline{f}] \text{ (n times)}.
\]
If we identify \( G \) with \( \mathbb{Z} \), the map \( h^* \) is the (unique) homomorphism from \( \mathbb{Z} \) to \( \mathbb{Z}/n\mathbb{Z} \) given by \( 1 \mapsto -n \). \( \square \)

3. Show that the map \( p : \mathbb{R}^2 \to S^1 \times S^1 \) given by \( p(\theta, \phi) = (e^{2\pi i \theta}, e^{2\pi i \phi}) \) is a covering map.

Proof. First, note that this map is continuous (by Theorem 19.6 for example, after composing with projections), and the surjectivity is clear since any element in the codomain can be written in terms of \( \theta \) and \( \phi \) as in the statement of the exercise. Let \( b \in S^1 \times S^1 \). We must show that there exists an open subset \( U \) of \( S^1 \times S^1 \) such that \( U \) is evenly covered by \( p \). Let us mimic the proof of Munkres Theorem 53.1 in proving the claim for \( b \neq ((1, 0), (1, 0)) \), in which case we can take \( U \) to be the set of all points \( (x, y) \) in \( S^1 \times S^1 \) such that neither of the angles of \( x \) and \( y \) in polar representation are integer multiples of \( 2\pi \), and then argue at the end that if we had chosen \( b = ((1, 0), (1, 0)) \) an analagous argument with a different \( U \) would work just as well. So, the above \( U \) is open as the complement of a one point set in a Hausdorff space. We must show that it is evenly covered by \( p \). The set \( p^{-1}(U) \) is the complement of the integer lattice in \( \mathbb{R}^2 \). As such, it is the disjoint union of patches of the form \( Q_{n,m} = (n, n + 1) \times (m, m + 1) \) where \( n, m \in \mathbb{Z} \). We will now consider a fixed \( n, m \) and show that the square \( Q = (n, n + 1) \times (m, m + 1) \) is mapped homeomorphically onto \( U \). The map is clearly a surjection: \( (e^{2\pi i \alpha}, e^{2\pi i \beta}) \), where \( 0 < \epsilon \leq \delta < 1 \), is the image of \( (n + \epsilon, m + \delta) \). If \( x, y \) are distinct in \( (n, n + 1) \), then \( e^{2\pi i x} \neq e^{2\pi i y} \) since equality would imply that \( x \) and \( y \) differ by an integer, but \( |x - y| < 1 \). So \( p \) is a bijection. It is continuous as the restriction of a continuous map. The closure of \( Q \) is the square \( [n, n + 1] \times [m, m + 1] \), which is compact, and hence \( p \) is a
homeomorphism by Theorem 26.6 onto the closure of $U$ which is $S^1 \times S^1$. The restriction then of $p|_{\overline{U}}$ to $Q$ is a homeomorphism, but this restriction is just $p|Q$. Thus, $p$ is a covering map. \hfill \Box

4. Use the covering map in the previous problem and Theorem 54.4 to show that $\pi_1(S^1 \times S^1; (1,1)) \cong \mathbb{Z} \times \mathbb{Z}$.

Proof. Since $\mathbb{R}^2$ is simply connected, the lifting correspondence $\Phi$ defined on page 345 (for $p$ as in the previous problem, $b_0 = (1,1)$, and $c_0 = (0,0)$) is a bijection

$\Phi : \pi_1(S^1 \times S^1) \rightarrow p^{-1}(b_0)$.

Also, $p^{-1}(b_0) = \mathbb{Z} \times \mathbb{Z}$, since it is precisely angles that are integer multiples of $2\pi$ that are angles of zero inclination from the positive $x$-axis (which is the inclination of each component of $b_0$). The bijective lifting correspondence becomes:

$\Phi : \pi_1(S^1 \times S^1) \rightarrow \mathbb{Z} \times \mathbb{Z}$.

We show that $\Phi$ is a homomorphism, and the theorem is proved. Given $[f]$ and $[g]$ in $\pi_1(S^1 \times S^1, b_0)$, let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths in $\mathbb{R}^2$ beginning at $c_0$. Let $(n_1, n_2) = \tilde{f}(1)$ and $(m_1, m_2) = \tilde{g}(1)$. Then $\Phi([f]) = (n_1, n_2)$ and $\Phi([g]) = (m_1, m_2)$. Let $\tilde{g}$ be the path

$\tilde{g}(s) = (n_1, n_2) + \tilde{g}(s)$

in $\mathbb{R}$. Because $p((n_1, n_2) + x) = p(x)$ for all $x \in \mathbb{R}^2$ (since $n_1$ and $n_2$ are integers), the path $\tilde{g}$ is a lifting of $g$. It begins at $(n_1, n_2)$. Since $f$ ends at the point at which $\tilde{g}$ begins, the product $\tilde{f} \cdot \tilde{g}$ is defined. Further,

$\tilde{p} \circ (\tilde{f} \cdot \tilde{g})(s) = \begin{cases} 
    \tilde{p} \circ \tilde{f}(2s) = f(2s) & \text{if } s \in [0, 0.5] \\
    \tilde{p} \circ \tilde{g}(2s - 1) = g(2s - 1) & \text{if } s \in [0.5, 1] 
\end{cases}$

which shows that $\tilde{f} \cdot \tilde{g}$ is a lifting of $f \cdot g$, and it begins at $(0,0)$ since $\tilde{f}$ begins at $(0,0)$. The endpoint of the path is

$\tilde{g}(1) = (n_1, n_2) + g(1) = (n_1, n_2) + (m_1, m_2) = (n_1 + m_1, n_2 + m_2)$.

It follows that

$\Phi([f] \cdot [g]) = (n_1 + m_1, n_2 + m_2) = \Phi([f]) + \Phi([g])$,

and $\Phi$ is an isomorphism. \hfill \Box