TOPOLOGY II

ABSTRACT. Further topics in point-set topology: local compactness, paracompactness, compactifications; metrizability; Baire category theorem; homotopy and the fundamental group. Topics may also include uniform spaces, function spaces, topological groups or topics from algebraic topology.

2. HOMEWORK 2

Problem 2.1. Given spaces $X$ and $Y$, let $[X,Y]$ denote the set of homotopy classes of maps of $X$ into $Y$.

(1) Let $I = [0,1]$. Show that for any $X$, the set $[X,I]$ has a single element.

Proof. Let $[f],[g] \in [X,I]$ where $f,g : X \to I$. Suppose, by way of contradiction that $f \neq g$. So there does not exist a continuous $F : X \times I \to I$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for each $x$. But observe that $F : X \times I \to I$ defined by $F(x,t) = (1-t)f(x) + tg(x)$ is a continuous map with $F(x,0) = f(x)$ and $F(x,1) = g(x)$. This is a contradiction to our hypothesis that there did not exist such a map. Conclude that $f \sim g$ which means $[f] = [g]$ and hence $[X,I]$ is trivial. The key here is that $I$ is convex and we have the straight line homotopy. \qed

(2) Show that if $Y$ is path connected, the set $[I,Y]$ has a single element.

Proof. Let $[f],[g] \in [I,Y]$ where $f,g : I \to Y$. Suppose, by way of contradiction, that $f \neq g$. So there does not exist a continuous $F : I \times I \to Y$ with $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for each $x$. Observe that $f \sim e_{f(0)}$ where $e_{f(0)} : I \to Y$ defined by $e_{f(0)}(t) = f(0)$ using the homotopy $F : I \times I \to Y$ defined by $F(s,t) = f(st)$ noting that $F(s,0) = f(s \cdot 0) = f(0)$ and $F(s,1) = f(s \cdot 1) = f(s) = f(s)$ and $F$ is continuous being the composition of two continuous functions. Similarly we can show that $g \sim e_{g(0)}$. Now since $Y$ is path connected, there exists a path $p$ from $f(0)$ to $g(0)$ which induces a homotopy $G : I \times I \to Y$ between $e_{f(0)}$ and $e_{g(0)}$ defined by $G(s,t) = p(t)$ which is continuous and $G(s,0) = p(0) = g(0)$ and $G(s,1) = p(1) = g(0)$. Conclude that $e_{f(0)} \sim e_{g(0)}$ and by transitivity of homotopy classes $f \sim e_{f(0)} \sim e_{g(0)} \sim g$ which means $[f] = [g]$ a contradiction to our assumption. Conclude that $[I,Y]$ is trivial.

As a side note: My first approach was to define $F(x,s) = p_x(s)$ where $p_x$ is a path from $f(x)$ to $g(x)$, whose existence we are assured of since $Y$ is path connected. The problem was that I couldn’t show $F$ was continuous defined like that and I suspect it might not be, since we don’t restrict how $p_x$ is defined, we just know it exists
and then we use it. So I ended up deciding to shrink each continuous function to a point and then consider the path between the two points.

As another side note, given a representative of an equivalence class in \([I, Y]\), it is a path, so the reasoning above could have easily used path homotopies instead of just homotopies. \(\square\)

**Problem 2.2.** A subset \(A\) of \(\mathbb{R}^n\) is said to be **star convex** if for some point \(a_0\) of \(A\), all the line segments joining \(a_0\) to other points of \(A\) lie in \(A\).

1. Find a star convex set that is not convex.

   **Proof.** Let’s work in \(\mathbb{R}^2\). Define \(A = \{(x, y) : y \leq |x|\}\) and choose \(a_0 = (0, 0)\). Observe that this is star convex: pick \((x, y) \in A\), then \(y \leq |x|\), if \(t \in [0, 1]\), then \((1 - t)a_0 + ty = ty = |t|y \leq |t||x| = |tx|\). If we consider \((-1, 1)\) and \((1, 1)\), it is obvious that \((1 - t)(-1, 1) + t(1, 1) \notin A\) for all \(t\), in particular, consider \(t = 1/2\): \((0, 1) \notin A\) since \(1 \leq 0\). \(\square\)

2. Show that if \(A\) is star convex, \(A\) is simply connected.

   **Proof.** Let \(A\) be star convex with \(a_0\) being our central point. \(A\) is path connected since if we pick two points \(x, y \in A\), we can find the straight line path \(f\) mapping \(x\) to \(a_0\) and the straight line path \(g\) mapping \(a_0\) to \(y\). Then \(f * g\) is a path from \(x\) to \(y\).

It remains to show that \(\pi_1(A, a_0) \cong \mathbb{Z}\).

Let \(f : I \to A\) be a loop based at \(a_0\).

If \(s_0 \in I\), then \((1 - t)f(s_0) + ta_0\) is a straight line segment connecting \(f(s_0)\) to \(a_0\).

Since \(A\) is star convex, then for any values \(t_0 \in I\) and \(s_0 \in I\), \((1 - t_0)f(s_0) + t_0 a_0 \in A\).

Hence \(H : I \times I \to A\) given by

\[
H(s, t) = (1 - t)f(s) + t a_0
\]

is well-defined, \(H\) is continuous since \(f\) is continuous, scalar multiplication is continuous and vector addition is continuous.
1. \( H(s, 0) = f(s) + 0a_0 = f(s) \)
2. \( H(s, 1) = (1-s) f(s) + 1a_0 = a_0 \)
3. \( H(0, t) = (1-t) f(0) + ta_0 = (1-t)a_0 + ta_0 = a_0 \)
4. \( H(1, t) = (1-t) f(1) + ta_0 = (1-t)a_0 + ta_0 = a_0 \)

By 1, 2, 3, 4 and the continuity and well definedness of \( H \), \( f \sim_p E a_0 \).

Since \( f \) was arbitrary and path homotopy equivalence is transitive, then any two loops in \( A \) based at \( a_0 \) are path homotopic.

Hence \( \pi_1(A, a_0) \cong \mathbb{Z} \). \( \square \)
3. Munkres §52.2.

Let \( \alpha \) be a path in \( X \) from \( x_0 \) to \( x_1 \); let \( \beta \) be a path in \( X \) from \( x_1 \) to \( x_2 \). Prove: If \( \gamma = \alpha \ast \beta \), then \( \dot{\gamma} = \dot{\beta} \circ \dot{\alpha} \).

Proof. Using the definitions of reverse, \( \dot{\alpha} \), \( \dot{\beta} \) and together with the well definedness of \( \ast \) we have,

\[
\gamma ([f]) = [\gamma] \ast [f] \ast [\gamma] = [\alpha \ast \dot{\beta}] \ast [f] \ast [\alpha \ast \dot{\beta}] = [\beta] \ast \dot{\alpha} \ast [f] \ast [\alpha] \ast [\beta] = [\beta] \ast \dot{\alpha} ([f]) \ast [\beta] = \dot{\beta} (\dot{\alpha} ([f])) = (\dot{\beta} \circ \dot{\alpha}) ([f]).
\]

Problem 2.4. Show that if \( X \) is path connected, the homomorphism induced by a continuous map is independent of base point, up to isomorphisms of the groups involved. More precisely, let \( h : X \to Y \) be continuous, with \( h(x_0) = y_0 \) and \( h(x_1) = y_1 \). Let \( \alpha \) be a path in \( X \) from \( x_0 \) to \( x_1 \), and let \( \beta = h \circ \alpha \). Show that

\[
\dot{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \dot{\alpha}.
\]

This equation expresses the fact that the following diagram of maps "commutes."

\[\text{[Diagram]}\]

\[\text{Equation}\]

\[\text{Comment}\]

\[\text{Footnote}\]

\[\text{Footnote 2}\]

\[\text{Footnote 3}\]
Proof. Since $X$ is path connected, we have $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ by Corollary 52.2 which gives us the isomorphism we want. Our continuous function and our path $\alpha$ induce another path in $Y$ which give us another isomorphism: $\pi_1(Y, y_0) \cong \pi_1(Y, y_1)$, by Theorem 52.1.

Observe that
\[ \hat{\beta}((h x_0)_*([f])) = \hat{\beta}([h \circ f]) = [\hat{\beta} \ast (h \circ f) \ast \beta] = [(h \circ \alpha) \ast (h \circ f) \ast (h \circ \alpha)] \]
and
\[ (h x_1)_*(\hat{\alpha}([f])) = (h x_1)_*([\alpha \ast f \ast \alpha]) = [h \circ (\alpha \ast f \ast \alpha)] = [(h \circ \alpha) \ast (h \circ f) \ast (h \circ \alpha)]. \]

Now just notice that $h \circ \alpha = h \circ \alpha$ since
\[ h \circ \alpha(s) = h \circ \alpha(1 - s) = h(\alpha(1 - s)) = h(\alpha(s)). \]