Topology - Homework 1

(1) Munkres §51 #1

Given topological spaces $X, Y,$ and $Z,$ suppose $h, h' : X \to Y$ are homotopic, witnessed by the homotopy

$$ H : X \times I \to Y $$

(where $H(s, 0) = h(s)$), and $k, k' : Y \to Z$ are homotopic, witnessed by

$$ K : Y \times I \to Z $$

(where $K(s, 0) = k(s)$). Let $h_t : X \to Y$ be defined by $h_t(x) = H(x, t),$ and define the map $G : X \times I \to Z$ by

$$ G(x, t) = K(h_t(x), t). $$

We verify that for any $x \in X,$

$$ G(x, 0) = K(h_0(x), 0) = K(h(x), 0) = (k \circ h)(x), $$

and

$$ G(x, 1) = K(h_1(x), 1) = K(h'(x), 1) = (k' \circ h')(x). $$

Further, if we set $F(x, t) = (H(x, t), \pi_2(x, t)),$ then $F$ is a continuous map by Theorem 19.6. Thus, $G = K \circ F$ is continuous. Thus $G$ is a homotopy witnessing that $k \circ h$ and $k' \circ h'$ are homotopic.

(2) Munkres §51 #3

(a) Let $H : \mathbb{R} \times I \to \mathbb{R}$ by

$$ H(x, t) = (1 - t)x. $$

If $f(t) = 1 - t,$ $g(x, t) = (\pi_1(x, t), f(\pi_2(x, t)))$, and $h$ is multiplication, then $f, g, h$ are continuous, and $H = h \circ g$ is continuous. Further, for all $x \in X,$ we have $H(x, 0) = x = i_X(x),$ and $H(x, 1) = 0.$ Thus $H$ witnesses that $i_X$ is homotopic to the constant map with image $\{0\}.$ $H$ is therefore null-homotopic and $\mathbb{R}$ is contractible.
Now let $H : I \times I \to \mathbb{R}$ by

$$H(x,t) = (1-t)x.$$  

We observe that for all $x \in I$, $H(x,0) = x = i(x)$ and $H(x,1) = 0$. Furthermore $H$ is continuous by an argument analogous to the one above. Thus $i$ is homotopic to a constant map and $I$ is contractible.

(b) Suppose $X$ is contractible. Let $H : X \times I \to X$ witness this, with $H(x,0) = \text{Id}(x)$, for all $x \in X$. Let $a$ and $b$ be two points in $X$. Let

$$G(t) = \begin{cases} 
H(b, 2t), & \text{if } t \in [0, \frac{1}{2}] \\
H(a, 2 - 2t), & \text{if } t \in [\frac{1}{2}, 1]
\end{cases}$$

Note that $H(b, 2t)$ is continuous, its domain of $[0, \frac{1}{2}]$ is closed, $H(a, 2 - 2t)$ is continuous, while $[\frac{1}{2}, 1]$ is also closed, and $H(b, 2t)$ agrees with $H(a, 2 - 2t)$ at $t = \frac{1}{2}$, which is the only element of $[0, \frac{1}{2}) \cap [\frac{1}{2}, 1]$. By the pasting lemma, $G$ is a continuous function from $[0, 1]$ to $X$. Further, $G(0) = H(b, 0) = b$, while $G(1) = H(a, 0) = a$, so that $G$ is a path from $b$ to $a$. Since $a, b$ were arbitrary in $X$, $X$ is path connected.

(3) Let $X$ be a topological space and let $P$ be the set of all paths in $X$. Show that the relation given by “$f \simeq g$ if and only if $f$ is path-homotopic to $g$” is an equivalence relation on $P$.

Reflexive: Let $f : I \to X$ be a path from $a$ to $b$. Define $F : I \times I \to X$ by $F(s,t) = f(s)$. Then $F$ is continuous since $f$ is by assumption, and also $F(s,t) = f(s)$ for $t = 0, 1$ (all $t$), while $F(0, t) = f(0) = a$ and $F(1, t) = f(1) = b$. So $F$ is a path homotopy between $f$ and $f$.

Symmetric: Letting $f$ be as above, let $g$ be another path from $a$ to $b$, and suppose there is a path homotopy $F$ between $f$ and $g$ such that $F(s, 0) = f(s)$ and $F(s, 1) = g(s)$. Let $G : X \times I \to X$ be the map defined by $G(s, t) = F(s, 1 - t)$. $G$ is continuous since $F$ is and by arguments similar to those in the previous problems, and we have $G(s, 0) = F(s, 1) = g(s), G(s, 1) = F(s, 0) = f(s), G(0, t) = F(0, 1 - t) = a$, and finally, $G(1, t) = F(1, 1 - t) = b$. Thus, $G$ is a path homotopy from $g$ to $f$ and $\simeq$ is symmetric.

Transitive: Let $f$, $g$, and $F$ be as above ("Symmetric" part), and let $h$ be another path from $a$ to $b$, with $g \simeq h$, witnessed by $H$ (i.e. $H(s, 0) = g(s), H(s, 1) = h(s)$). Define $K$ as follows:

$$K(s,t) = \begin{cases} 
F(s, 2t), & \text{if } t \in [0, \frac{1}{2}] \\
H(s, 2t - 1), & \text{if } t \in [\frac{1}{2}, 1]
\end{cases}$$

$K$ is well defined at $t = \frac{1}{2}$ (equal to $g(s)$ there) and the hypotheses of the pasting lemma are satisfied so that $K$ is a continuous function. Further, $K(s, 0) = ...$
$F(s, 0) = f(s)$, while $K(s, 1) = H(s, 1) = h(s)$. Also, for all $t \in [0, \frac{1}{2}]$, $K(0, t) = F(0, 2t) = a$, and $K(1, t) = F(1, 2t) = b$; and for all $t \in [\frac{1}{2}, 1]$, $K(0, t) = H(0, 2t - 1) = a$, $K(1, t) = H(1, 2t - 1) = b$, so that for all $t \in I$, $K(0, t) = a$ and $K(1, t) = b$. It follows that $K$ is a path-homotopy between $f$ and $h$ so that $\simeq$ is transitive and, in conclusion, an equivalence relation.