Pasting lemma

Let $X$ be a top space and $A, B$ be closed sets in $X$ s.t. $X = A \cup B$. Suppose $f : A \to Y$ and $g : B \to Y$ are continuous maps s.t. $f(x) = g(x)$ for all $x \in A \cup B$. Then $h : X \to Y$ s.t. $h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$ is continuous.

**Proof**

Let $C$ be a closed set in $Y$.

$$h^{-1}(C) = \{ x \in X \mid h(x) \in C \}$$

$$= \{ x \in A \mid f(x) \in C \} \cup \{ x \in B \mid g(x) \in C \}$$

$$= f^{-1}(C) \cup g^{-1}(C)$$

By continuity of $f$ and $g$, $f^{-1}(C)$ is closed in $A$ and $g^{-1}(C)$ is closed in $B$. 

Announcements

- Exam a week from today
By Lemma 17.2 in Munkres,
\[ f^{-1}(c) \text{ is closed in } A \text{ iff } \exists K_A \subseteq X \text{ a closed set s.t. } f^{-1}(c) = K_A \cap A. \]
Similarly \[ g^{-1}(c) = K_B \cap B \text{ for } K_B \text{ closed in } X. \]

Hence \[ h^{-1}(c) = (K_A \cap A) \cup (K_B \cap B). \]
Since finite intersections and finite unions of closed is closed, then \( h^{-1}(c) \) is closed.

Hence \( h \) is continuous. \( \Box \)

**Box and Product topologies**

Recall: Given top. spaces \((X, \mathcal{E}_X)\) and \((Y, \mathcal{E}_Y)\) the topology on \( X \times Y \) is generated by basis \( \mathcal{B} \) \( \mathcal{E}_X \times \mathcal{E}_Y \).

**Motivating Question:** What topologies make sense on infinite products \( \prod_{\alpha \in J} X_\alpha \)

\[ \prod_{\alpha \in J} X_\alpha = \{ \tilde{x} : J \rightarrow \bigcup_{\alpha \in J} X_\alpha \bigm| \tilde{x}(\alpha) \in X_\alpha \} \]

\[ \tilde{x} \in \prod_{\alpha \in J} IR \quad \tilde{x} = \{ x_\alpha \}_{\alpha \in J} \text{ or } \tilde{x} : J \rightarrow IR \]
Def: Let \( \{ X_\alpha \}_{\alpha \in J} \) be a collection of top. spaces.

The box topology on \( \prod_{\alpha \in J} X_\alpha \) has basis

\[
\mathcal{B}_{\text{box}} = \left\{ \prod_{\alpha \in J} U_\alpha \mid U_\alpha \text{ is open in } X_\alpha \right\}
\]

Exercise: Check this is a basis.

Def: The product topology on \( \prod_{\alpha \in J} X_\alpha \) has basis

\[
\mathcal{B}_{\text{prod}} = \left\{ \prod_{\alpha \in J} U_\alpha \mid U_\alpha \text{ is open in } X_\alpha \text{ and } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \in J \right\}
\]

Exercise: Check that this is a basis.

Remarks:

1. \( \mathcal{B}_{\text{box}} \subset \mathcal{B}_{\text{prod}} \) so box topology is finer than the product topology.
2. For finite products the Box and Product topologies are the same.
3. Alternate interpretation of the product top:

   Topology with sub-basis

   \[
   S = \left\{ \prod_{\alpha \in J} (U_\alpha^{-1}(U_\alpha)) \mid \alpha \in J \text{ and } U_\alpha \subset X_\alpha \text{ is open} \right\}
   \]

   Where \( \prod_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha \) is the projection.

Why? Finite intersections \( \prod_{\beta = 1}^m (U_{\alpha_i}) \) are exactly the elements of \( \mathcal{B}_{\text{prod}} \).
Facts | If $X_\alpha$ has basis $B_\alpha$ for each $\alpha \in \mathcal{J}$,
then $\left\{ \prod_{\alpha \in \mathcal{J}} B_\alpha | B_\alpha \in B_\alpha \right\}$ is a basis for the box topology.

Similarly $\left\{ \prod_{\alpha \in \mathcal{J}} B_\alpha | B_\alpha \in B \text{ for finitely many } \alpha \right\}$
is a basis for the product topology.

Ex. | $\prod_{i=1}^\infty (0,1) \subset \prod_{i=1}^\infty \mathbb{R}$ is open in the box topology
but is not open in the product topology.

Prop | If $\{X_\alpha\}_{\alpha \in \mathcal{J}}$ is a collection of Hausdorff spaces, then $\prod_{\alpha \in \mathcal{J}} X_\alpha$ is Hausdorff in both the prod. and box topologies.

Pf | Exercise.

Prop | Let $A_\alpha \subset X_\alpha$ for each $\alpha \in \mathcal{J}$. Let $\prod_{\alpha \in \mathcal{J}} A_\alpha$ have the product (box) topology. Then the product topology on $\prod_{\alpha \in \mathcal{J}} A_\alpha$ agrees with the subspace topology on $\prod_{\alpha \in \mathcal{J}} A_\alpha \subset \prod_{\alpha \in \mathcal{J}} X_\alpha$. 
Prop. If $A_\alpha \subseteq X_\alpha$, then in either the box or product topology, $\prod A_\alpha = \prod A_\alpha$.

The following is one reason why we prefer the product topology.

Prop. Let $f: Y \to \prod_{x \in \mathcal{Y}} X_x$ where $\prod X_\alpha$ has the product topology. Then $f$ is continuous if each $f_\beta = \prod_\beta f$ of $f: Y \to X_\beta$ is continuous.

Proof. Let $f_\beta = \prod_\beta f$. Note that $\prod_\beta$ is continuous (as proved last time) and $f$ is continuous by assumption. Hence $f_\beta$ is continuous since the composition of continuous maps is continuous.

Suppose each $f_\beta$ is continuous. We will show that $f^{-1}$ (any sub-basis element) is open. This implies $f$ is continuous. Let $\prod_\beta^{-1}(U_\beta)$ be a sub-basis for the product topology.

$$f^{-1}(\prod_\beta^{-1}(U_\beta)) = (\prod_\beta f)^{-1}(U_\beta) = f_\beta^{-1}(U_\beta)$$

Hence $f^{-1}(\prod_\beta^{-1}(U_\beta))$ is open by continuity of $f_\beta$. □
Is false for the product box topology.

Example: \( \prod_{i=1}^{\infty} \mathbb{R} \) with the box topology.

Let \( f: \mathbb{R} \rightarrow \prod_{i=1}^{\infty} \mathbb{R} \) be \( f(t) = (t, t, \ldots) \)

\( f_i: \mathbb{R} \rightarrow \mathbb{R} \) s.t. \( f(x) = x \) is continuous for each \( i \).

However \( B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \ldots \) is open in \( \prod_{i=1}^{\infty} \mathbb{R} \).

\( f^{-1}(B) = \{ x \in \mathbb{R} | x \in (-\frac{1}{n}, \frac{1}{n}) \text{ for every } n \in \mathbb{N}^+ \} \)

\( = \{ 0 \} \leftarrow \text{not open.} \)