Recall: $Y \subseteq X$ is closed iff $X - Y$ is open.

Key facts about closed sets

Property: Let $X$ be a topological space

1. $X$ and $\emptyset$ are closed sets
2. Arbitrary intersections of closed sets are closed.
3. Finite intersections of closed sets are closed.

Proof:

1. $\emptyset$ is open and $X = X - \emptyset$.
2. $X$ is open and $\emptyset = X - X$.
3. Suppose $\{A_\alpha\}_{\alpha \in \mathcal{J}}$ is a collection of closed sets.
   
   Examine $X - \left( \bigcap_{\alpha \in \mathcal{J}} A_\alpha \right) = \bigcup_{\alpha \in \mathcal{J}} (X - A_\alpha)$, by de Morgan's Law.
   
   Since $A_\alpha$ is closed, $X - A_\alpha$ is open for each $\alpha \in \mathcal{J}$.
   
   Since $X - A_\alpha$ is open for each $\alpha \in \mathcal{J}$, then $\bigcup_{\alpha \in \mathcal{J}} (X - A_\alpha)$ is open.
   
   Since $\bigcup_{\alpha \in \mathcal{J}} (X - A_\alpha)$ is open, then $\bigcap_{\alpha \in \mathcal{J}} A_\alpha$ is closed. \(\square\)

2. Similar to 1).
Q: How do closed subsets relate to the subspace topology?

**Prop.** Let \( A \subseteq Y \subseteq X \) where \( X \) is a top. space. \( A \) is a closed set in \( Y \) with the subspace topology iff \( A = Y \cap C \) where \( C \) is closed in \( X \).

**Pf.** \( \Rightarrow \)

Suppose \( A \) is closed in \( Y \).

Thus, \( Y - A \) is open in \( Y \).

By def. of subspace top., \( Y - A = Y \cap U \) where \( U \) is open in \( X \).

Hence, \( X - U \) is closed.

Claim: \( A = Y \cap (X - U) \)

**Exercise**

\( \Leftarrow \)

Suppose \( C \) is closed in \( X \) and \( A = Y \cap C \).

Thus, \( X - C \) is open in \( X \).

By def. of subspace top, \( Y \cap (X - C) \) is open in \( Y \). Hence, \( Y - (Y \cap (X - C)) \) is closed in \( Y \).

Claim: \( A = Y - (Y \cap (X - C)) \)

**Exercise**
Closure

Def: Given \( A \subseteq X \), the closure of \( A \), denoted \( \overline{A} \), is the intersection of all closed sets that contain \( A \).

Facts:
- \( \overline{A} \) is closed
- \( A \subseteq \overline{A} \)
- \( A = \overline{A} \) iff \( A \) is closed (exercise)
- \( \overline{A} \) is the "smallest" closed set containing \( A \)
- \( \overline{X} = X \), \( \overline{\emptyset} = \emptyset \).

The following is a useful alternative characterization.

Prop: Let \( A \subseteq X \). \( x \in \overline{A} \) iff every open set \( U \) that contains \( x \) intersects \( A \) non-trivially.

Pic:

Ex: \( a \) \in \( \mathbb{R} \), \( a \in (a, b) \)

PF: (Instead of "\( P \Leftrightarrow Q \)", we will show "not \( P \Leftrightarrow \) not \( Q \).")

Show \( x \notin \overline{A} \) iff there exists \( U \subseteq X \) open s.t. \( U \cap A = \emptyset \).
\[ \Rightarrow \] If \( x \notin \overline{A} \), then \( x \in X - \overline{A} \). Since \( \overline{A} \) is closed, \( X - \overline{A} \) is open. Thus \( X - \overline{A} \) is an open set that contains \( x \) and is disjoint from \( A \).

\[ \Leftarrow \] Suppose there exists an open set \( U \subset X \) s.t. \( x \in U \) and \( U \cap A = \emptyset \). Since \( U \) is open \( X - U \) is closed. Since \( \overline{A} \) is the intersection of all closed sets containing \( A \), and \( x \notin X - U \), then \( x \notin \overline{A} \). \( \square \)

**Useful Prop** | If \((X, \mathcal{B})\) has basis \( \mathcal{B} \), then \( x \in \overline{A} \) iff every basis element containing \( x \) intersects \( A \).

**Examples** | Let \( IR_s \) be \( IR \) with the standard topology.

- In \( IR_s \), \( A = \{ \frac{1}{n} | n \in \mathbb{Z}^+ \} \), \( \overline{A} = A \cup \{0\} \)
- In \( IR_s \), \( \emptyset = \emptyset \).
- In \( IR_s \), \( A = (a, b) \), \( \overline{A} = [a, b] \)

**Note:** There is something special about points in \( \overline{A} - A \).

**Def** | Let \( X \) be a top. space and \( A \subset X \). A point \( x \in X \) is a limit point of \( A \) if every open set \( U \) that contains \( x \) intersects \( A - \{x \} \) non-trivially. (Equivalently, \( x \) is a limit point of \( A \) if \( x \in \overline{A} - \{x \} \).)
Examples
- In $\mathbb{R}_s$, the limit points of $[0,1]$ are all points in $[0,1]$.
- In $\mathbb{R}_s$, the limit points of $\left\{ \frac{1}{n} \mid n \in \mathbb{Z}^+ \right\}$ is just 0.
- In $\mathbb{R}_s$, the limit points of $\{0\}$ is $\emptyset$.

Let $A'$ denote the set of limit points of a set $A$.

**Theorem**

$\overline{A} = A \cup A'$.

**Proof**

**$\subseteq$** Let $x \in \overline{A}$

Case 1: If $x \in A$ then $x \in A \cup A'$

Case 2: If $x \notin A$, then by prop. every open set containing $x$ intersects $A$ nontrivially. Since $x \in A$, then every open set containing $x$ intersects $A - \{x\}$ nontrivially. So $x \notin A'$

In either case $x \in A \cup A'$.

**$\supseteq$** First, $A \subset \overline{A}$ by def. of closure

Claim: $A' \subset A$

Let $x \in A'$, then every open set that contains $x$ intersects $A - \{x\}$ nontrivially. So, every open set that contains $x$ and intersects $A$ non trivially, by previous prop. $x \in \overline{A}$.

Hence $A \cup A' \subseteq \overline{A}$.