Topology 26

Announcements
- Final exam a week from today
- Next time Admittance for 550 B.

Def | Given $x, y \in X$ a path in $X$ from $x$ to $y$ is a continuous map $f: [0,1] \rightarrow X$ with $f(0) = x$ and $f(1) = y$. $X$ is path connected if any two points can be joined by a path.

Lemma | If $X$ is path connected, then $X$ is connected.

Pf | Suppose $X$ is path connected and has a separation $X = A \cup B$. Pick $a \in A$ and pick $b \in B$.

Since $X$ is path connected there exists a continuous function $f: [0,1] \rightarrow X$ with $f(0) = a$ and $f(1) = b$.

Recall $[0,1]$ is connected and the continuous image of connected is connected. Hence, $f([0,1])$ is connected.

By a previous lemma, $f([0,1]) \subseteq A$ or $f([0,1]) \subseteq B$. Either conclusion contradicts the fact that $ANB = \emptyset$, since $a \in A$ and $b \in B$. □
Prop: If \( A \) is a connected subspace of \( X \) and \( A \subseteq B \subseteq \overline{A} \), then \( B \) is connected.

Proof: Suppose \( B \) has a separation \( B = C \cup D \).
Since \( A \) is connected, then by Munkres 23.21
\( A \subseteq C \) or \( A \subseteq D \), which say \( A \subseteq C \).
Hence, \( \overline{A} \subseteq \overline{C} \) and \( B \subseteq \overline{C} \). Thus
\[ \text{Inc} \overline{C} \neq \emptyset. \] This contradicts Munkres 23.1 which says \( \text{Inc} \overline{C} = \emptyset \). So, \( B \) has no separation. \( \square \)

Key example

"Topologist's sine curve"


\[ F: (0, 1] \rightarrow \mathbb{R}^2, \quad F(t) = (t, \sin(\frac{1}{t})) \]

Claim: \( F((0, 1]) \) is connected.

Proof: This follows from the fact that \( (0, 1] \) is p.c.o.
and \( F \) is continuous.

Claim: \( \overline{F((0, 1])} = F((0, 1]) \cup (\mathbb{Q} \times [-1, 1]) \)

Proof: exercise.
Examples of path connected spaces
- \( \mathbb{R}^n \)
- \( \mathbb{R}^n = \{ x \in \mathbb{R}^n | \|x\| \leq 1 \} \)
- Any "convex" subset of \( \mathbb{R}^n \)
- \( \mathbb{R}^n - \{ \vec{0}_n \} \) for \( n \geq 2 \).

**Lemma** The continuous image of a path connected space is path-connected.

**Proof** Let \( X \) be path connected and let \( f: X \to Y \) be a continuous map. Let \( a, b \in f(X) \).

Pick \( x \in f^{-1}(a) \) and \( y \in f^{-1}(b) \).
Since \( X \) is path connected there is a path \( g: [0, 1] \to X \) from \( x \) to \( y \). Since \( f \) and \( g \) are continuous, then \( f \circ g: [0, 1] \to Y \) is a path from \( a \) to \( b \).

**Claim** \( S^n \) for \( n \geq 1 \) is path connected.

\[ f: \mathbb{R}^n - \{ \vec{0}_n \} \to S^{n-1}, \quad f(x) = \frac{x}{\|x\|} \]

is a continuous map. Since \( \mathbb{R}^n - \{ \vec{0}_n \} \) is p.c. for \( n \geq 2 \) then \( S^k \) is p.c. for \( k \geq 1 \). \( \square \)
Note, by our previous proof, $F((0,1])$ is connected.

Claim: $F((0,1])$ is not path connected.

Proof: Suppose, to form a contradiction, that there is a path from some point $a\in\mathbb{S}^2\times[-1,1]$ to some point $b$ in $F((0,1])$. We can choose to represent $g$ as a parametrization $g(t) = (x(t), y(t))$. WLOG we can assume $x(t) > 0$ for $t > 0$.

Hence, $y(t) = \sin\left(\frac{1}{x(t)}\right)$ for $t > 0$.

Note $\frac{1}{x(t)} \to 0$ as $n \to \infty$.

For each $n > 1$ there exists $u_n \in (0, x(\frac{1}{n}))$ such that $\sin\left(\frac{1}{u_n}\right) = (-1)^n$.

Now $x$ is continuous, so $\exists t_n \in (0, \frac{1}{n})$ s.t. $x(t_n) = u_n$.

Thus $y(t_n) = \sin\left(\frac{1}{x(t_n)}\right) = (-1)^n$.

Since $t_n \to 0$ but $y(t_n)$ does not converge, we have a contradiction to continuity of $y$. □
Recall: A path in $X$ is a continuous map $f : [0,1] \to X$.

**Def.** Given $x_0 \in X$, a loop in $X$ based at $x_0$ is a path $f : [0,1] \to X$ s.t. $f(0) = f(1) = x_0$.

We want a natural notion of equivalence of based loops.

**Def.** Two loops based at $x_0 \in X$ are homotopic if there exist a continuous function

$$F : [0,1] \times [0,1] \to X \text{ s.t. } F(0,t) = f(t) \text{ and } F(1,t) = g(t).$$

**Ex.**

$$f(t) = \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix}, \quad g(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$F(s,t) = \begin{pmatrix} (1-s)\cos(2\pi t) + s \\ (1-s)\sin(2\pi t) \end{pmatrix}$
There is a binary operation on based loops called concatenation (or stacking). If $f$ and $g$ are loops based at $x_0$, then $f * g : [0,1] \rightarrow X$

$$f * g (t) = \begin{cases} g(2t) & 0 \leq t \leq \frac{1}{2} \\ f(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Let $Y$ be the set of loops based at $x_0$. Let $\sim$ be the equivalence relation on $Y$ that $f \sim g$ if $f$ is homotopic to $g$.

Claim: $Y/\sim$ under the binary operation of stacking is a group!

Recall group axioms:
1. Closure (f * g is still a loop based at $x_0$)
2. Identity element (the constant loop is the identity)
3. Inverse element (f(1-t) is the inverse of f(t))
4. Associativity (f * (g*h) = (f*g)*h).
$Y/n$ is denoted $\pi_1(X, x_0)$ and is known as the fundamental group of $X$ based at $x_0$.

Examples:

$\pi_1(\mathbb{R}^2, (0,0)) \cong \mathbb{Z}$

$\pi_1(S^1, (1,0)) \cong \mathbb{Z}$

$\pi_1(S^1 \times S^1, (1,0) \times (1,0)) \cong \mathbb{Z} \times \mathbb{Z}$

$\pi_1(S^2_a, \circ) \cong \pi_1(D^2) \cong \mathbb{Z}$

$\pi_1(S^2_{a^2=1}, \circ) \cong \pi_1(\mathbb{R}P^2) \cong \langle a \rangle \langle a^2=1 \rangle$

This is called a group presentation

\[
\left\langle \frac{a}{\text{generators}} \middle| \frac{a^2=1}{\text{relations}} \right\rangle \cong \mathbb{Z}_2
\]
The free group on two variables.

So, \( \pi_1 ( \bigvee_{b=1}^{n} S') \cong F_n \) (free group on \( n \) generators).

**Def.** A group is finitely presented if it can be constructed from a finite list of generators and a finite list of relations.

\[
\langle a, b, c \mid aba^{-1}b^{-1} = 1, c^3 = 1 \rangle
\]

\[
\pi_1 ( \bigvee_{a,b,c} U \bigvee_{b^{-1}} a^{-1} U \bigvee_{c} c ) \cong \langle a, b, c \mid aba^{-1}b^{-1} = 1, c^3 = 1 \rangle
\]

Topology contains all group theory.

2-dimensional CW complexes can all be embedded in \( \mathbb{R}^5 \). So, subspaces of \( \mathbb{R}^5 \) contain all group theory.