Recall Lemma: If $X$ is a top space, and $Y$ is a set and $p : X \to Y$ is surjective, then $\exists!$ topology on $Y$ s.t. $p : X \to Y$ is a quotient map.

(Define $V \subset Y$ to be open if $p^{-1}(V)$ is open)

Given an equivalence relation $\sim$ on $X$ there is a natural quotient topology to define on $X/\sim$ (the set of equivalence classes of $\sim$ on $X$).

Be careful with Quotient maps and topologies

Fact: If $p : X \to Y$ is a quotient map and $A \subset X$ is a subspace, $p|A$ need not be a quotient map.

Example: We have seen that $\pi_1 : X \times Y \to X$ s.t. $\pi_1(x,y) = x$ is a quotient map.

Let $A = \{(0,0)\} \cup \{(x, y) | x \neq 0, y < 1\}$.

Examine $p|A$.

Note $\pi_1(0,0) = A$ is open in $A$.

But, $\pi_1(0) = 1 \in A$ is not open.

Since $\left(\pi_1|A\right)^{-1}(0) = \pi_1(0,0) \subset A$ this contradicts $p$ is a quotient map.
**Fact** If \( p_1 : X_1 \to Y_1 \) and \( p_2 : X_2 \to Y_2 \) are quotient maps, then \( p_1 \times p_2 : X_1 \times X_2 \to Y_1 \times Y_2 \) need not be a quotient map.

(pg. 143 Example 7)

**Lemma** If \( p : X \to Y \) and \( q : Y \to Z \) be quotient maps, then \( q \circ p : X \to Z \) is a quotient map.

**Pf** Suppose \( V \subset Z \) is open wts \( (q \circ p)^{-1}(V) \) is open.

\[
(q \circ p)^{-1}(V) = p^{-1}(q^{-1}(V))
\]

Since \( q \) is a quotient map \( q^{-1}(V) \) is open in \( Y \).
Since \( p \) is \( p^{-1}(q^{-1}(V)) \) is open in \( X \).
Hence, \( (q \circ p)^{-1}(V) \) is open.

Suppose \( (q \circ p)^{-1}(V) \) is open.

\[
p^{-1}(q^{-1}(V)) \text{ is open.}
\]

Since \( p \) is a quotient map \( q^{-1}(V) \) is open.
Since \( q \) is a quotient map \( V \) is open.

Thus \( q \circ p \) is a quotient map.
**Countability Axioms**

**Motivation:** If \((x, d)\) is a metric space with the metric topology and \(x \in X\) then \(\exists\) a countable collection of nbhs of \(x\), \(\bigcup_{i=1}^{\infty} U_i\), s.t. Any nbh of \(x\) contains one of the \(U_i\).

**Def:** A top. space \(X\) is **first-countable**, if given any \(x \in X\), \(\exists\) a countable collection of nbhs of \(x\) given by \(\bigcup_{i=1}^{\infty} U_i\), s.t. any nbh of \(x\) contains some \(U_i\).

**Prop:** If \(X\) is first-countable

a) Given \(A \subseteq X\), \(x \in \overline{A}\) iff \(\exists\) a seq. \(\{x_n\} \in A\) s.t. \(x_n \to x\).

b) Given \(f: X \to Y\), \(f\) is cont. iff \(\forall x_n \to x\) in \(X\), we have \(f(x_n) = f(x)\).

**Proof:** Same as for metric spaces

**Def:** A top. space \(X\) is **2nd-countable** if \(\exists\) a basis \(B\) for \(X\) s.t. \(B\) is countable

**Note:** Second Countable \(\Rightarrow\) first-countable

(Let \(U_n\) be the collection of all basis elements that contain \(x\).)
Ex] $\mathbb{R}$ with the standard topology is 2nd-countable.
Let $B = \{(a, b) \mid a < b \text{ and } a, b \in \mathbb{Q}\}$
$B$ is a basis for $\mathbb{R}$. (Hw 0.1)
Why is $B$ countable?
$P : B \to \mathbb{Q} \times \mathbb{Q}$ s.t. $P$ is 1-1.
$P(\{a, b\}) = (a, b)
$if $(a_1, b_1) = (a_2, b_2)$ as points in $\mathbb{Q} \times \mathbb{Q}$
then $(a_1, b_1) = (a_2, b_2)$ as open intervals.
Hence, $B$ is in bijection with a subset of a countable set, so, $B$ is countable.

Ex] $\mathbb{R}_{\infty}$ is not second-countable. Let $B$ be any basis.
Recall $[x, x+1)$ is open in $\mathbb{R}_{\infty}$ for all $x$.
Hence $\exists B_x \in B$ s.t. $x \in B_x < [x, x+1)$
However, if $x \neq y$ then $B_x \neq B_y$.
(Since $\inf(B_x) = x \neq y = \inf(B_y)$)
So, $B$ has uncountably many elements.

Ex] A metric space need not be 2nd-countable.
Let $\mathbb{R}$ have the discrete metric.
Since any basis for $\mathbb{R}$ must contain the singletons.