Th \textsuperscript{m} (Heine-Borel)  

A subspace $K \subset \mathbb{R}^n$ is compact iff $K$ is closed and bounded.

**Pf**  
Suppose $K \subset \mathbb{R}^n$ is compact. Since $\mathbb{R}^n$ is Hausdorff, then $K$ is closed.

Suppose to form a contradiction that $K$ is unbounded. Then \( \exists \cup_k \forall x \in K \neq \emptyset \) s.t.

\[ U_k = (-k, k) \times (-k, k) \times \cdots \times (-k, k) \]  

is an open cover of $K$ with no finite sub-cover. Hence $K$ must be bounded.

\[ \Leftarrow \]  
Suppose $K$ is closed and bounded.

Since $K$ is bounded there exists a closed box \( B = [-L, L] \times \cdots \times [-L, L] \) s.t. \( K \subset B \).

Since the product of compact spaces is compact, then $B$ is compact.

Since closed subsets of compact spaces are compact then $K$ is compact. (Here we are implicitly using the H.W. problem that states subspace top. on $K \subset \mathbb{R}^n$ is same as $K \subset B \subset \mathbb{R}^n$.)
Theorem (Extreme Value thm)

Let \( f: X \to \mathbb{R} \) be continuous, with \( X \) compact. Then \( f \) takes on its maximum and minimum values.

Proof: Since the continuous image of compact is compact, then \( f(X) \) is a compact subset of \( \mathbb{R} \). By Heine Borel, \( f(X) \) is closed and bounded.

Let \( M = \sup(f(X)) \). Since \( f(X) \) is bounded, \( M < \infty \). By def. of supremum \( \exists \{a_n\}_{n=1}^{\infty} \) contained in \( f(X) \) s.t. \( \lim_{n \to \infty} a_n = M \).

Since \( f(X) \) is closed \( f(X) = f(X) \) and \( M \in f(X) \). Thus \( f \) attains its maximum. A similar argument shows \( f \) attains its minimum.


Other notions of compactness

Definition: \( X \) is limit point compact if every infinite subset of \( X \) has a limit point.

Example: \( \mathbb{R} \) is not l.p. compact since \( \mathbb{Z} \subset \mathbb{R} \) has no limit points.

Proposition: If \( X \) is compact then \( X \) is limit point compact.

Proof: Exercise
\[ \text{Ex} \quad X = (\mathbb{Z}_+, \text{discrete}) \times (\mathbb{R}, \text{indiscrete}) \]

\[ \text{Claim} \quad X \text{ is not compact.} \]

Look at the open cover \( U_n = \times n \times Y \)

\[ b \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \]

\[ \text{Claim} \quad X \text{ is l.p. compact} \]

\[ \text{Pf} \quad \text{Let } A \text{ be any non empty set.} \]

Since \( A \text{ is non-empty } \exists n \in \mathbb{N}_+ \text{ s.t. } (n, a) \text{ or } (n, b) \in A. \]

If \((n, a) \in A\), then \((n, b)\) is a limit point of \(A\). If \((n, b) \in A\), then \((n, a)\) is a limit point of \(A\).

\[ \text{Def} \quad X \text{ is sequentially compact if every sequence} \]

in \(X\) has a convergent subsequence.

In general, not comparable to compactness.

\[ \text{Thm} \quad \text{If } X \text{ is metrizable, TFAE} \]

1) \(X\) is compact
2) \(X\) is limit point compact
3) \(X\) is sequentially compact

\[ \text{Pf} \quad \text{See real analysis text.} \]
Def Let $X$ be a top. space. A collection $\mathcal{C}$ of subsets of $X$ has the finite intersection property if for every finite subcollection $\mathcal{C}_1, \ldots, \mathcal{C}_n \subseteq \mathcal{C}$, the intersection $\bigcap_{i=1}^n \mathcal{C}_i \neq \emptyset$.

Thm $X$ is compact iff every collection of closed sets $\mathcal{C}$ with the finite intersection property also has the property $\bigcap_{\mathcal{C} \in \mathcal{C}} \mathcal{C} \neq \emptyset$.

Ex $X = (0, 1]$ and $\mathcal{C} = \{ (0, \frac{1}{n}) \mid n \in \mathbb{Z}^+ \}$. The sets in $\mathcal{C}$ are closed in $X$ and have the finite intersection property. However $\bigcap_{n \in \mathbb{N}^+} (0, \frac{1}{n}) = \emptyset$.

pf First some observations

Let $\mathcal{U}$ be any collection of sets in $X$ and $\mathcal{C} = \{ X - U \mid U \in \mathcal{U} \}$.

1. Elements of $\mathcal{U}$ are open $\iff$ elements of $\mathcal{C}$ are closed
2. $\mathcal{U}$ covers $X$ $\iff$ $\bigcap_{\mathcal{C} \in \mathcal{C}} \mathcal{C} = \emptyset$
3. $\{ U_1, \ldots, U_n \} \subseteq \mathcal{U}$ covers $X$ $\iff$ $\bigcap_{i=1}^n U_i = \emptyset$. 


(Contrapositive of the definition of compactness)

X is compact iff “for any collection \( U \) of open sets, if no finite subcover of \( U \) covers \( X \) then \( U \) does not cover \( X \).”

This statement is equivalent to

“for any collection \( \mathcal{C} \) of closed sets, if every finite intersection is non-empty, then \( \bigwedge_{C \in \mathcal{C}} C \neq \emptyset \)”

Corollary: If \( X \) is compact and \( C_1 \supseteq C_2 \supseteq \ldots \)

is a sequence of nested non-empty closed sets, then \( \bigwedge_{n=1}^{\infty} C_n \neq \emptyset \).

Proof: Let \( \mathcal{C} = \{C_n\} \subseteq \mathcal{P}(X) \). \( \mathcal{C} \) has the finite intersection property, so \( \bigwedge_{n=1}^{\infty} C_n = \emptyset \).