(1) Munkres §30 Exercise 5
   (a) Every metrizable, separable space has a countable basis.

   **Proof**: Suppose $X$ is metrizable and $Q$ is a countable dense subset of $X$. Let $d$ be a metric on $X$ that induces the topology with which $X$ is equipped. Let $n \in \mathbb{Z}^+$, and for each $x \in Q$, let $B_n(x)$ be the ball of radius $\frac{1}{n}$ centered at $x$. Since the set $A_n = \{B_n(x) | x \in Q\}$ is indexed by $Q$, $A_n$ is countable. Let $\mathscr{B} = \bigcup_{n \in \mathbb{Z}^+} A_n$. Then $\mathscr{B}$ is countable as the countable union of countable sets. The claim is that $\mathscr{B}$ is basis for the topology on $X$.

   First, let $x \in X$. Fix $n > 0$. Since $Q$ is dense in $X$, there exists a $q$ in $Q$ such that $d(x, q) < \frac{1}{n}$; that is $x \in B_n(q) \in \mathscr{B}$. This verifies the first basis axiom.

   Second, suppose $B_n(x)$ and $B_m(y)$ are two members of $\mathscr{B}$ that have a nonempty intersection $I$. Let $z \in I$. Choose $k$ so that

   \[
   \frac{1}{k} < \min \left\{ \frac{1}{n} - d(x, z), \frac{1}{m} - d(y, z) \right\},
   \]

   and choose $p \in Q$ so that $z \in B_k(p)$ (since $Q$ is dense). It will be shown that $B_k(p) \subset I$.

   Let $q \in B_k(p)$. We have

   \[
   d(q, x) < d(q, p) + d(p, x) < d(q, p) + d(p, z) + d(z, x) < \frac{1}{k} + \frac{1}{k} + d(z, x) < 2 \frac{1}{k} + d(z, x) < \min \left\{ \frac{1}{n} - d(x, z), \frac{1}{m} - d(y, z) \right\} + d(x, z) \leq \frac{1}{n}.
   \]

   Hence, $q \in B_n(x)$. Similar computations show that $q \in B_m(y)$. Thus, $q \in I$, and one can conclude that $B_k(p) \subset I$, verifying the second basis axiom. □

   (b) Every metrizable Lindelof space has a countable basis.

   **Proof**: Let $X$ be metrizable and Lindelof. Let $d$ be a metric on $X$ that induces the topology with which $X$ is equipped. For each $n \in \mathbb{Z}^+$, $x \in X$, let $B_n(x)$ be defined as in (a), and observe that these balls cover $X$, so since $X$ is Lindeloff, there exists a countable subcollection $A_n$ of these balls that also covers $X$. Let $\mathscr{B} = \bigcup_{n \in \mathbb{Z}^+} A_n$, which is countable as the countable union
of countable sets. If \( x \in X \), then for any \( n \in \mathbb{Z}^+ \), \( A_n \) covers \( X \), so there is some ball in \( A_n \), and hence in \( \mathcal{B} \), that contains \( x \). Next, suppose \( x \in B_n(x_1) \cap B_m(x_2) \) for two elements \( x_1, x_2 \) in \( X \). Let \( \frac{1}{k} < \frac{1}{2} \min \left\{ \frac{1}{n} - d(x, x_1), \frac{1}{m} - d(x, x_2) \right\} \). Let \( x_0 \in X \) such that \( x \in B_k(x_0) \). Let \( y \in B_k(x_0) \) be arbitrary. Then we have

\[
d(y, x_1) \leq d(y, x_0) + d(x, x_0) + d(x, x_1)
\]

\[
\leq \frac{2}{k} + d(x, x_1)
\]

\[
\leq \frac{1}{n}.
\]

This implies that \( B_k(x_0) \subset B_n(x_1) \). Similar computations show that \( B_k(x_0) \subset B_m(x_2) \), and hence \( B_k(x_0) \subset B_n(x_1) \cap B_m(x_2) \), so that \( \mathcal{B} \) is in fact a basis. \( \square \)

(2) Prove that if \( X \) has unique limits and is first countable, then \( X \) is Hausdorff.

\textbf{Proof:} Assume the given hypotheses for \( X \). Let \( x \) and \( y \) be distinct points in \( X \). Let \( A = \{ A_n \mid n \in \mathbb{Z}^+ \} \) and \( B = \{ B_n \mid n \in \mathbb{Z}^+ \} \) be countable bases at \( x \) and \( y \) respectively, with the proviso that \( B_n \neq B_m \) if \( n \neq m \). The latter proviso is admissible, since one can remove redundant elements from a countable basis without changing the countability or the status of being a basis. Let \( n_1 = 1 \). Having chosen \( n_1, n_2, \ldots, n_j \), let \( n_{j+1} \) be the least positive integer not included in \( \{ n_1, n_2, \ldots, n_j \} \) such that \( A_{n_{j+1}} \subset A_{n_j} \), if such an integer exists; otherwise let \( n_{j+1} = n_j \). This furnishes a decreasing (with respect to set inclusion) subsequence \( \{ A_{n_j} \} \) of \( A \). Construct an analogous subsequence \( \{ B_{m_j} \} \) of \( B \). For each positive integer \( j \), let \( I_j = A_{n_j} \cap B_{m_j} \). Since \( X \) is not Hausdorff, each \( I_j \) is nonempty, so choose an element \( x_j \in I_j \). Let \( U \) be an open subset of \( X \) that contains \( x \). Since \( A \) is a countable basis at \( x \), there exists a positive integer \( j \) such that \( A_{n_j} \subset X \). Then since \( X \supset A_{n_j} \supset A_{n_{j+1}} \supset A_{n_{j+2}} \supset \cdots \), it follows that \( U \) contains all \( x_k \) for \( k \geq j \). Since \( U \) was an arbitrary open set, it follows that \( \{ x_k \} \) converges to \( x \). Let \( V \) be an open subset of \( X \) that contains \( y \). Since \( B \) is a countable basis at \( y \), there exists a positive integer \( l \) such that \( B_{n_l} \subset X \). Then since \( X \supset B_{n_l} \supset B_{n_{l+1}} \supset B_{n_{l+2}} \supset \cdots \), it follows that \( U \) contains all \( x_k \) for \( k \geq l \). Since \( V \) was an arbitrary open set, it follows that \( \{ x_k \} \) converges to \( y \). This contradicts the uniqueness of limits. Hence, \( X \) is Hausdorff. \( \square \)

(3) Munkres §30 Exercise 4: Every compact metrizable space has a countable basis.

\textbf{Proof:} This follows from 5(b) above: Every compact space is Lindelof, since finite covers are in particular countable covers. But here is an argument carried out independently of 5(b).

Let \( X \) be a compact metrizable space with a countable basis. Since \( X \) is metrizable, let \( d \) be a metric on \( X \) that induces the topology with which \( X \) is equipped. Let \( n \in \mathbb{Z}^+ \) be fixed. To each \( x \in X \), let \( B(x; \frac{1}{n}) \) be the ball centered at \( x \) with radius \( \frac{1}{n} \). Then
\[ \bigcup_{x \in X} B(x; \frac{1}{n}) \text{ contains } X \text{ as a subset, since in particular for each } x_0 \in X, x_0 \in B(x_0, \frac{1}{n}). \] Thus, \( \{B(x; \frac{1}{n}) | x \in X\} \) is an open cover of \( X \). Since \( X \) is compact, the latter cover admits a finite subcover, so there exists a finite subset \( A \subset X \) such that \( A_n = \{B(x; \frac{1}{n}) | x \in A\} \) is a cover of \( X \). Since \( n \) was arbitrary in \( \mathbb{Z}^+ \), one can construct such a collection \( A_n \) for each \( n \in \mathbb{Z}^+ \). The claim is that \( \mathcal{B} = \bigcup_{n \in \mathbb{Z}^+} A_n \) is a basis for the topology on \( X \). By proving the claim, one proves the theorem, since \( \mathcal{B} \) is countable as a countable union of finite sets.

Let \( x \in X, n \in \mathbb{Z}^+ \). Since \( A_n \) is a cover of \( X \), there exists some \( x_0 \in X \) such that \( x \in B(x_0; \frac{1}{n}) \). Since \( B(x_0; \frac{1}{n}) \in \mathcal{B} \), this verifies one of the two basis axioms. For the second one, recycle "\( n \)" and suppose \( B(x; \frac{1}{n}) \) and \( B(y; \frac{1}{m}) \) are two members of \( \mathcal{B} \) that have a nonempty intersection \( I \). Let \( z \in I \). Choose \( k \) so that

\[
\frac{1}{k} < \min \left\{ \frac{\frac{1}{n} - d(x, z)}{2}, \frac{\frac{1}{m} - d(y, z)}{2} \right\},
\]

and choose \( p \in X \) so that \( z \in B(p; \frac{1}{k}) \). It will be shown that

\[ B(p; \frac{1}{k}) \subset I. \]

Let \( q \in B(p; \frac{1}{k}) \). We have

\[
d(q, x) < d(q, p) + d(p, x) < d(q, p) + d(p, z) + d(z, x) < \frac{1}{k} + \frac{1}{k} + d(z, x) < \frac{2}{k} + d(z, x) < \frac{1}{n} - d(x, z), \frac{1}{m} - d(y, z) \right\} + d(z, x) \leq \frac{1}{n},
\]

Hence, \( q \in B(x; \frac{1}{n}) \). Similar computations show that \( q \in B(y; \frac{1}{m}) \).

Thus, \( q \in I \), and one can conclude that \( B(p; \frac{1}{k}) \subset I \), verifying the second basis axiom. \( \square \)