Topology Homework 1

February 4, 2014

Solutions

§13#8

(a) Let $\mathcal{B} = \{ (a, b) | a < b \text{ and } a, b \in \mathbb{Q} \}$. Let $U$ be open in $\mathbb{R}$ (with the standard topology), and let $x \in U$. Since $U$ is a union of basis elements (by Lemma 13.1), there exists an interval $(a, b), a, b \in \mathbb{R}$, such that $x \in (a, b) \subset U$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists c, d \in \mathbb{Q}$ with $a < c < x < d < b$. Thus, $(c, d) \in \mathcal{B}$, with $x \in (c, d) \subset U$. It follows from Lemma 13.2 that $\mathcal{B}$ is a basis for the standard topology on $\mathbb{R}$.

(b) Let $\mathcal{B} = \{ [a, b) | a < b, a, b \in \mathbb{Q} \}$. First we show $\mathcal{B}$ is a basis. Let $x \in \mathbb{R}$. For any rational $a < x$ and rational $b > x$, we have $x \in [a, b) \in \mathcal{B}$. This verifies the first condition. Next, let $B_1 = [a, b), B_2 = [c, d)$ be two intervals in $\mathcal{B}$ with a nonempty intersection that contains $x$. Letting $e$ be the greater of $a$ and $c$ and letting $f$ be the lesser of $b$ and $d$, we obtain $x \in B_1 \cap B_2 = [e, f) \in \mathcal{B}$. This verifies the second condition (using $B_1 \cap B_2 = B_3$). Thus, $\mathcal{B}$ is a basis. Call the topology it generates $T$. Now if $U \in T$, then by Lemma 13.1, $U$ is the union of elements of $\mathcal{B}$, and since every rational is also real, $U$ is also the union of certain basis elements for the lower limit topology. By 13.1 again, $U$ is a member of the lower limit topology. Thus, the lower limit topology is finer than $T$. Now observe that $\pi \in [\pi, 4] \subset \mathbb{R}$. Suppose $\pi \in B = [a, b) \in \mathcal{B}$. Since $a$ is rational, it must be that $\pi > a$. But then $B \not\subset [\pi, 4]$. Thus there does not exist a basis element in $\mathcal{B}$ that contains $\pi$ and is a subset of $[\pi, 4)$. By Lemma 13.3, $T$ is not finer than the lower limit topology. The lower limit topology is then strictly finer than $T$, and therefore unequal to $T$.

§16#1

Let $A, Y,$ and $X$ be as given in the problem. Let $T_Y$ be the topology that $A$ inherits from $Y$. Let $T_X$ be the topology that $A$ inherits from $X$. Then

$$T_Y = \{ A \cap W | W \text{open in } Y \}$$

$$T_X = \{ A \cap W | W \text{open in } X \}.$$
Let $U \in T_Y$. There exists $W$ in $Y$ such that $U = A \cap W$. Also, since $Y$ is a subspace of $X$, there exists a $V$ open in $X$ such that $W = X \cap Y$. Then $U = A \cap (V \cap Y) = A \cap V$, since $A \subset Y$. Thus, $U \in T_X$. It follows that $T_Y \subset T_X$. Now let $U \in T_X$. There exists a $W$ open in $X$ such that $U = A \cap W$. Also $W \cap Y$ is open in $Y$, and since $A \subset Y$, $U = A \cap W = A \cap (W \cap Y)$. Thus $A \in T_Y$, and it follows that $T_X \subset T_Y$. Thus $T_X = T_Y$.

§16#4

Let $U$ be open in $X \times Y$. Let $T_X$ and $T_Y$ be the topologies on $X$ and $Y$ respectively. Then

$$\pi_1(U) = \pi_1\left( \bigcup_{\alpha \in Z} A_\alpha \times B_\alpha \right)$$

$$= \bigcup_{\alpha \in Z} \pi_1(A_\alpha \times B_\alpha)$$

where $Z$ and the open $A_\alpha$'s and $B_\alpha$'s are furnished by Theorem 13.1. From the above equality it is clear that $\pi_1(U) = \bigcup_{\alpha \in Z} A_\alpha$, which is open in $X$ as a union of open sets. Thus, $\pi_1$ is an open map. A symmetric argument establishes that $\pi_2$ is also an open map.

§16#9

Let $\mathcal{B}$ be the basis for the dictionary order topology $T$ on $\mathbb{R} \times \mathbb{R}$. Let $\mathcal{B}'$ be the basis for the product topology $T'$ on $\mathbb{R}_d \times \mathbb{R}$. Let $(x, y) \in \mathbb{R} \times \mathbb{R}$ and let $W \in \mathcal{B}$ such that $(x, y) \in W$. Then $W = ((a, b), (c, d))$ for some real $a, b, c, d$. As case 1, suppose $a = x$. Then $y > b$, and for some $c > y$ we have $(x, y) \in S \equiv \{x\} \times (b, c) \subset W$ and $S \in \mathcal{B}'$. As case 2, suppose $x = b$. Then $y < d$, and for some $f < y$ we have $(x, y) \in S \equiv \{x\} \times (f, d) \subset W$ and $S \in \mathcal{B}'$. Finally, as case 3, suppose $a < x < b$. Then for any real $z$ whatsoever, by virtue of the dictionary order we have $(a, b) < (z, z) < (c, d)$. Thus, letting $(e, f)$ be any nonempty real interval whatsoever, we get $(x, y) \in S \equiv \{x\} \times (e, f) \subset W$ and $S \in \mathcal{B}'$. Thus, in all cases we can find an element $S \in \mathcal{B}'$ such that $(x, y) \in S \subset W$. By Lemma 13.3, $T \subset T'$. Now let $(x, y) \in \mathbb{R} \times \mathbb{R}$ and let $U \times (c, d) \in \mathcal{B}$ such that $(x, y) \in U \times (c, d)$. Then $(x, y) \in ((x, c), (x, d)) \subset U \times (c, d)$. Since $((x, c), (x, d)) \in \mathcal{B}$, it follows from Lemma 13.3 that $T' \subset T$. Consequently $T' = T$. Consider the interval $I = ((a, b), (a, c)) \in \mathcal{B}$ in the dictionary order topology on $\mathbb{R} \times \mathbb{R}$. Let $x \in I$. Since the standard topology has as a basis all open rectangles, if one can show that $R \not\subset I$ for any open rectangle $R$ such that $x \in R$, then it will follow from Lemma 13.3 that the standard topology is not finer than $T' = T$. Let $R$ be an open rectangle containing $x$, i.e. $R = (r, s) \times (q, t)$. Suppose $R \subset I$. Then for any element $y$ in $R$ it must be that $y = (y_1, y_2)$ for $b < y_2 < c$. This is a contradiction since $y' = (a + \frac{y_2 - a}{2}, y_2 + \frac{t - y_2}{2})$ is an element in $R$, but $y' \not\in I$. So the standard topology is not finer than $T$. Next, recycle symbols and let $R = (a, b) \times (c, d)$ be a basis element for
the standard topology, and let \((x, y) \in R\). Then \(B = ((x, c), (x, d)) \in \mathcal{B}\), and \((x, y) \in B \subseteq R\). Thus, \(T\) is finer than the standard topology on \(\mathbb{R}^2\).