Math 500, Homework 7

Covering spaces, fundamental group

Due at start of class, Thursday, 12/8

Reading §53-55

Exercises (to do on your own)

1. Find a covering \( p : E \to B \) of the figure-eight \( B \) such that \( p^{-1}(b) \) consists of three points for each \( b \).

2. Let \( X \) be the figure-eight space, and let \( x_0 \) be the “crossing point” in the middle. Convince yourself (without rigorous proof) that \( \pi(X, x_0) \) is non-abelian. (This means there exist \([f],[g] \in \pi_1(X, x_0)\) such that \([f] \ast [g] \neq [g] \ast [f].\))

3. Let \( G \) and \( H \) be groups. Prove that the operation on \( G \times H \) defined by

\[
(g_1, h_1) \cdot (g_2, h_2) = (g_1 \cdot g_2, h_1 \cdot h_2)
\]

makes \( G \times H \) into a group.

4. Convince yourself that the Brouwer fixed point theorem could fail if you replaced \( B \) with the open disk \( B^o = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \).

Problems (to turn in)

1. Let \( p : E \to B \) be a covering map. Given \( b \in B \), prove that \( p^{-1}(\{b\}) \) is a discrete subspace of \( E \) (i.e., the subspace topology is the discrete topology).

2. Prove that a covering map \( p : E \to B \) is an open map. (Being also surjective and continuous, it follows that \( p \) is a quotient map.)

3. Let \( X \) and \( Y \) be spaces, with \( x_0 \in X \) and \( y_0 \in Y \). Prove that

\[
\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)
\]

as groups. (Exercise 3 explains the meaning of the product of two groups.)