Knot theory Day 9

Outline
- Kauffman Bracket Poly.
- Jones Poly.

<u>Def</u>] The Kauffman bracket Poly. of a link diagram D is a Laurent poly. $\langle D \rangle \in \mathbb{Z}[A^{\pm 1}]$ (i.e. polynomials in variables A and A^{-1}) defined by

0) $\langle D \rangle$ is invariant under R0

1) 

(called the skein relation)

2) $\langle D \amalg \mathcal{U} \rangle = (-A^2 - A^{-2}) \langle D \rangle$ where $\mathcal{U}$ is any closed crossingless loop in the diagram disjiont from D.

3) $\langle \mathcal{U} \rangle = 1$.

With these axioms, $\langle D \rangle$ produces a unique polynomial for any diagram

$\langle D \rangle$ is <u>not</u> a knot invariant.

<u>Ex</u> By def. $\langle O \rangle = 1$.

On the other hand

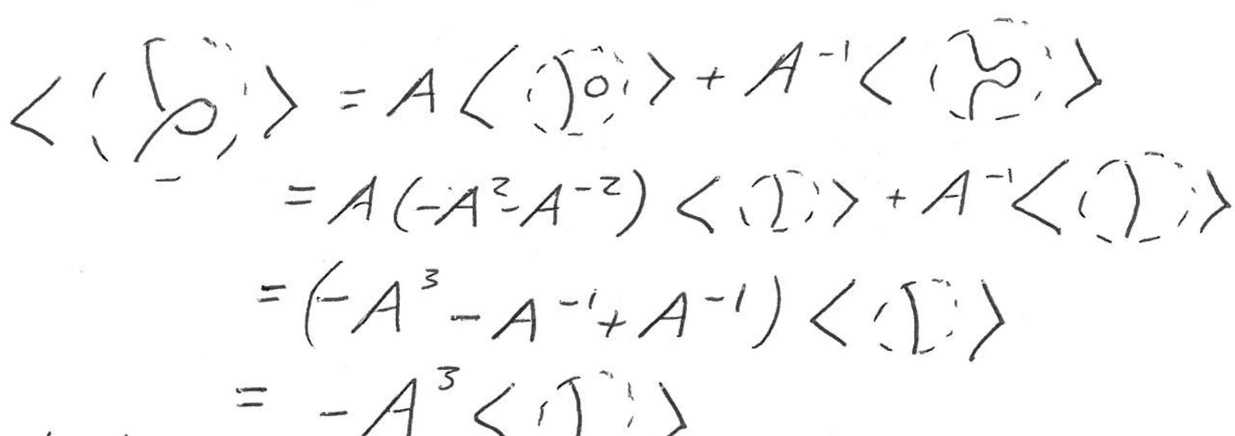
$$\langle \infty \rangle = A \langle O\,O \rangle + A^{-1} \langle \infty \rangle$$

$$= A(-A^2 - A^{-2}) \langle O \rangle + A^{-1} \langle O \rangle$$

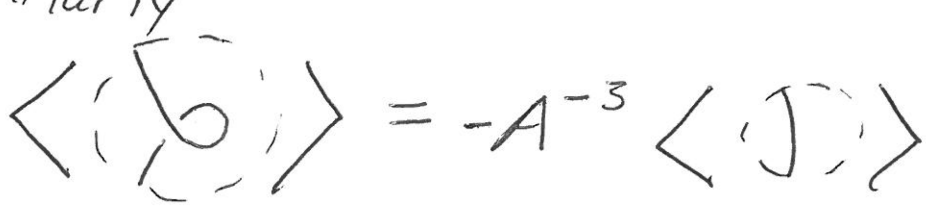
$$= -A^3 - A^{-1} + A^{-1}$$

$$= -A^3$$

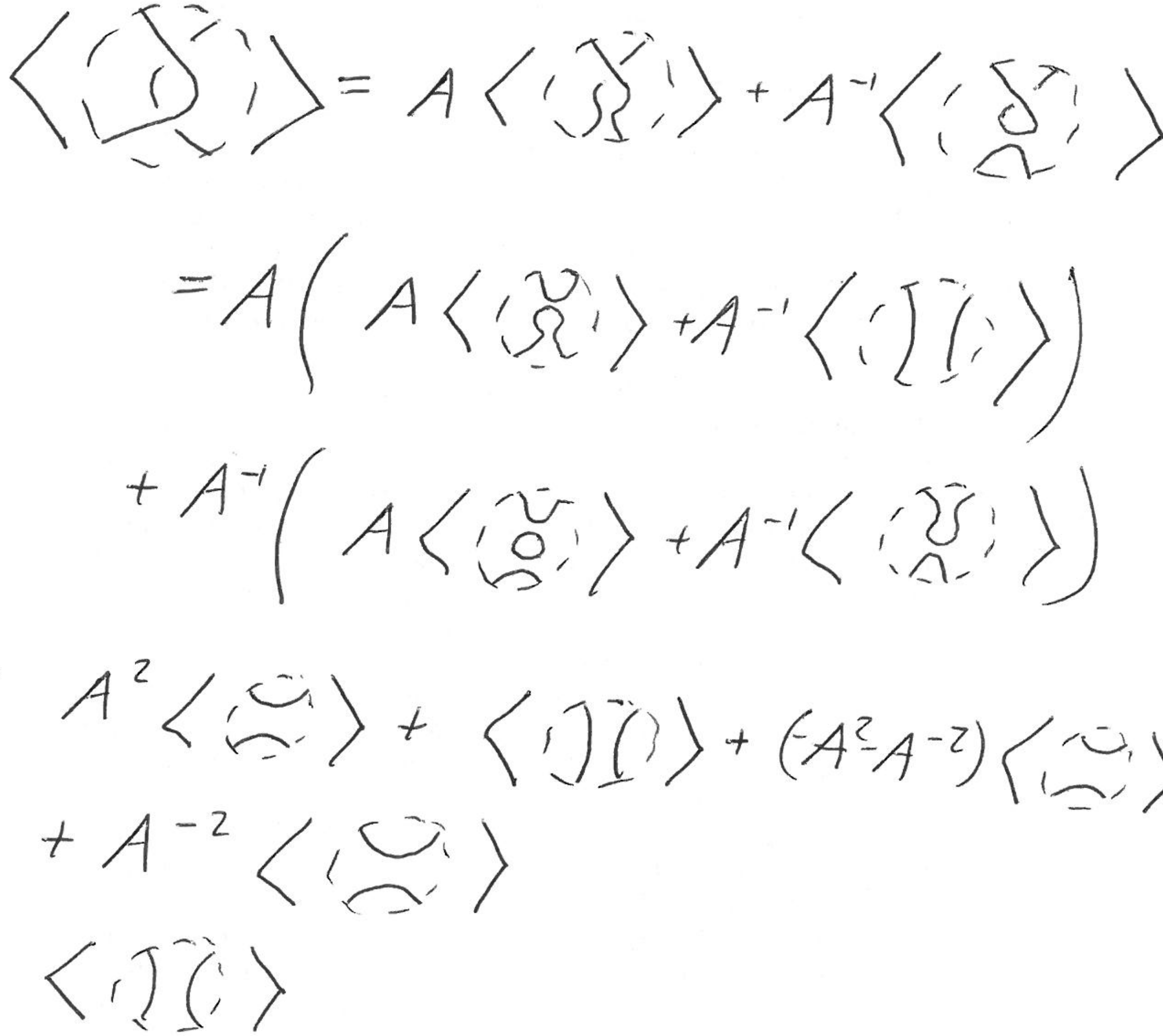
Claim: The Kauffman bracket is not invariant under the RI move.



Similarly



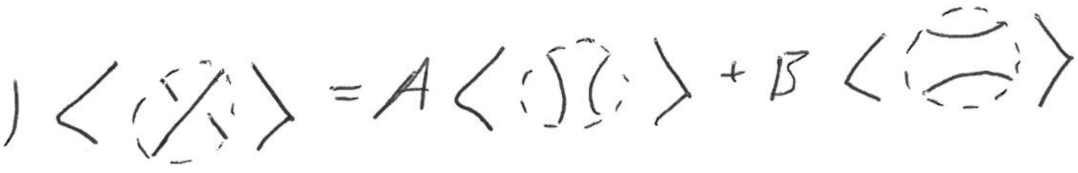
Claim: The Kauffman Bracket is invariant under the R II move



Claim: The Kauffman Bracket is invariant under the R III move

Proof) Exercise

Remark Suppose you have a definition of a polynomial derived from knot diagrams such that

1) 

2) $\langle D \amalg U \rangle = C \langle D \rangle$

3) $\langle U \rangle = 1$.

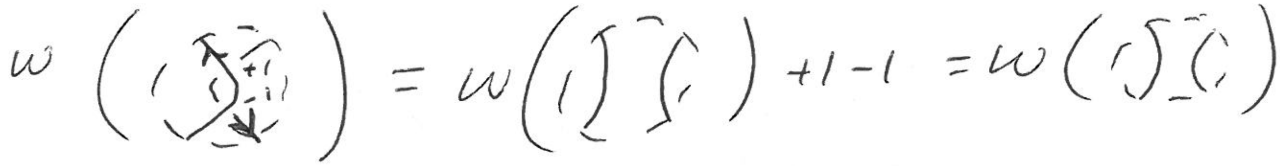
If this polynomial is invariant under $R\,II$ and $R\,III$ moves then it <u>must be</u> that $B = A^{-1}$ and $C = -A^2 - A^{-2}$.

Proof Exercise.

Using writhe to elevate $\langle D \rangle$ to a knot invariant

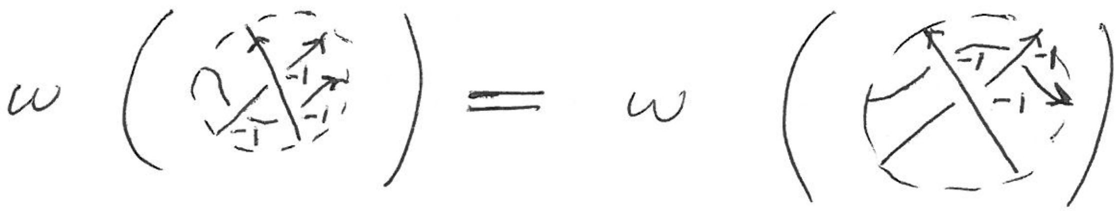
Lemma The writhe of an oriented link is invariant under $R\,II$ and $R\,III$ moves by changes by ± 1 under $R\,I$ moves.

Pf | RII



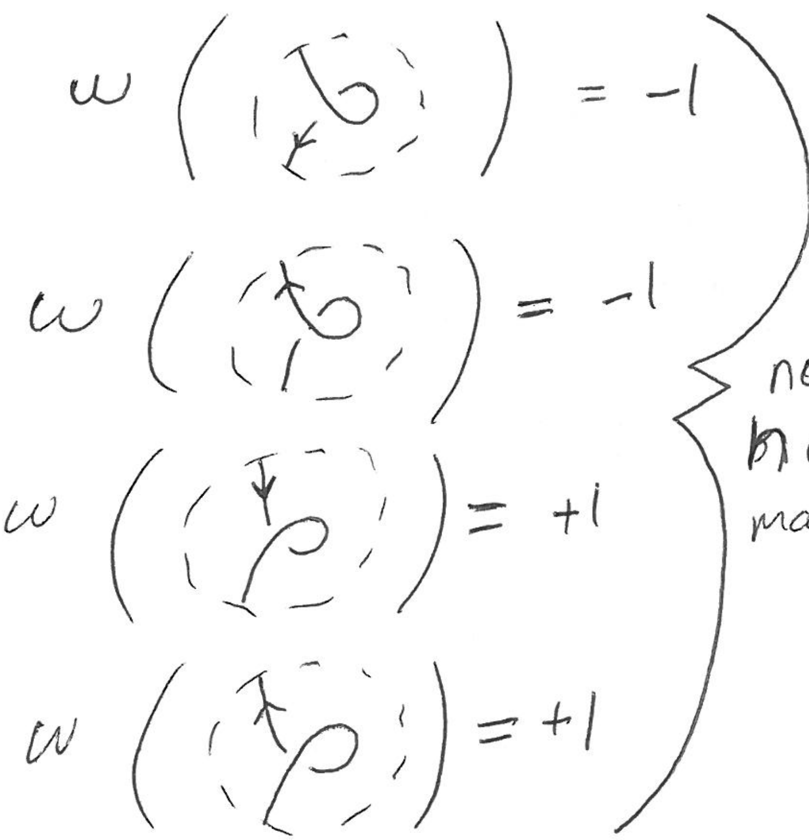
2 more cases to check

R III



7 more cases to check

R I



notice it is the crossing, not the orientation that matters.

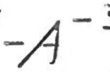
Thm If D is a knot diagram then

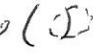
$$f_D(A) = (-A^3)^{-w(D)} \langle D \rangle$$ is

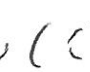
a knot invariant.

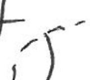
Pf Since $w(D)$ and $\langle D \rangle$ are invariant under $R\,II$ and $R\,III$, then $f_D(A)$ is invariant under $R\,II$ and $R\,III$.

f  $(A) = (-A^3)^{-w(}$  $^{)}$ $\langle$  $\rangle$

$= (-A^3)^{-(-1 + w(}$ $^{))} \cdot (-A^{-3}) \langle$  $\rangle$

$= (-A^3)(-A^{-3}) \cdot (-A^3)^{-w(}$  $^{)} \cdot \langle$  $\rangle$

$= -A^{-w(}$  $^{)} \cdot \langle$  $\rangle$

$= f$  (A)

Similarly f  $(A) = f$  (A). $\square$