

Knot theory Day 9

Outline

- Kauffman Bracket Poly.
- Jones Poly.

Def The Kauffman bracket Poly. of a link diagram D is a Laurent poly. $\langle D \rangle \in \mathbb{Z}[A^{\pm 1}]$ (ie. polynomials in variables A and A^{-1}) defined by

0) $\langle D \rangle$ is invariant under RO

$$1) \left\langle \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \right\rangle = A \left\langle \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right\rangle + A^{-1} \left\langle \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right\rangle$$

(called the skein relation)

2) $\langle D \sqcup U \rangle = (-A^2 - A^{-2}) \langle D \rangle$ where U is any closed crossingless loop in the diagram disjoint from D .

$$3) \langle U \rangle = 1.$$

With these axioms, $\langle D \rangle$ produces a unique polynomial for any diagram

$\langle D \rangle$ is not a knot invariant.

Ex | By def. $\langle O \rangle = 1$.

On the other hand

$$\begin{aligned}\langle \infty \rangle &= A \langle OO \rangle + A^{-1} \langle \infty \rangle \\ &= A(-A^2 - A^{-2}) \langle O \rangle + A^{-1} \langle O \rangle \\ &= -A^3 - A^{-1} + A^{-1} \\ &= -A^3\end{aligned}$$

Claim: The Kauffman bracket is not invariant under the RI move.

$$\begin{aligned}\langle \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \rangle &= A \langle \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \rangle + A^{-1} \langle \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \rangle \\ &= A(-A^2 - A^{-2}) \langle \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \rangle + A^{-1} \langle \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \rangle \\ &= (-A^3 - A^{-1} + A^{-1}) \langle \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \rangle \\ &= -A^3 \langle \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \rangle\end{aligned}$$

Similarly

$$\langle \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \rangle = -A^{-3} \langle \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \rangle$$

Claim: The Kauffman Bracket is invariant under the R II move

$$\begin{aligned}
 \langle \text{Diagram 1} \rangle &= A \langle \text{Diagram 2} \rangle + A^{-1} \langle \text{Diagram 3} \rangle \\
 &= A \left(A \langle \text{Diagram 4} \rangle + A^{-1} \langle \text{Diagram 5} \rangle \right) \\
 &\quad + A^{-1} \left(A \langle \text{Diagram 6} \rangle + A^{-1} \langle \text{Diagram 7} \rangle \right) \\
 &= A^2 \langle \text{Diagram 8} \rangle + \langle \text{Diagram 5} \rangle + (A^2 - A^{-2}) \langle \text{Diagram 9} \rangle \\
 &\quad + A^{-2} \langle \text{Diagram 10} \rangle \\
 &= \langle \text{Diagram 5} \rangle
 \end{aligned}$$

Claim: The Kauffman Bracket is invariant under the R III move

Proof Exercise

Remark Suppose you have a definition of a polynomial derived from knot diagrams such that

$$1) \langle \text{crossing} \rangle = A \langle \text{positive crossing} \rangle + B \langle \text{negative crossing} \rangle$$

$$2) \langle D \cup U \rangle = C \langle D \rangle$$

$$3) \langle U \rangle = 1.$$

If this polynomial is invariant under $R II$ and $R III$ moves then it must be that $B = A^{-1}$ and $C = -A^2 - A^{-2}$.

Proof Exercise.

Using writhe to elevate $\langle D \rangle$ to a knot invariant

Lemma The writhe of an oriented link is invariant under $R II$ and $R III$ moves by changes by ± 1 under $R I$ moves.

Pf | R II

$$w \left(\begin{array}{c} \text{diagram} \\ \text{crossing} \end{array} \right) = w \left(\begin{array}{c} \text{diagram} \\ \text{crossing} \end{array} \right) - 1 + 1 = w \left(\begin{array}{c} \text{diagram} \\ \text{crossing} \end{array} \right)$$

$$w \left(\begin{array}{c} \text{diagram} \\ \text{crossing} \end{array} \right) = w \left(\begin{array}{c} \text{diagram} \\ \text{crossing} \end{array} \right) + 1 - 1 = w \left(\begin{array}{c} \text{diagram} \\ \text{crossing} \end{array} \right)$$

2 more cases to check
R III

$$w \left(\begin{array}{c} \text{diagram} \\ \text{crossing} \end{array} \right) = w \left(\begin{array}{c} \text{diagram} \\ \text{crossing} \end{array} \right)$$

7 more cases to check

R I

$$w \left(\begin{array}{c} \text{diagram} \\ \text{crossing} \end{array} \right) = -1$$

$$w \left(\begin{array}{c} \text{diagram} \\ \text{crossing} \end{array} \right) = -1$$

$$w \left(\begin{array}{c} \text{diagram} \\ \text{crossing} \end{array} \right) = +1$$

$$w \left(\begin{array}{c} \text{diagram} \\ \text{crossing} \end{array} \right) = +1$$

notice it is the crossing, not the orientation that matters.

Thm | If D is a knot diagram then

$$f_D(A) = (-A^3)^{-w(D)} \langle D \rangle \text{ is}$$

a knot invariant.

Pf | Since $w(D)$ and $\langle D \rangle$ are invariant under R_{II} and R_{III} , then $f_D(A)$ is invariant under R_{II} and R_{III} .

$$\begin{aligned} f_{\textcircled{G}}(A) &= (-A^3)^{-w(\textcircled{G})} \langle \textcircled{G} \rangle \\ &= (-A^3)^{-(-1+w(\textcircled{J}))} \cdot (-A^{-3}) \langle \textcircled{J} \rangle \\ &= (-A^3)(-A^{-3}) \cdot (-A^3)^{-w(\textcircled{J})} \langle \textcircled{J} \rangle \\ &= -A^{-w(\textcircled{J})} \cdot \langle \textcircled{J} \rangle \\ &= f_{\textcircled{J}}(A) \end{aligned}$$

Similarly $f_{\textcircled{P}}(A) = f_{\textcircled{J}}(A)$. \square