Knot Theory Day 7

- Outline
- unknotting number
- crossing number
- bridge number
- width

Unknotted Number

Def: The unknotting number of a knot $K$ is the minimal number of crossing changes necessary over all diagrams $\gamma$ of $K$ to change $K$ to the unknot. (denoted $u(K)$).

Ex: $T_2$

\[ u(T_2) = 1 \]
**Def** The crossing number of $K$ is the minimal number of crossings in any diagram of $K$. (denoted $c(K)$)

**Ex.**

\[
\begin{array}{c}
\text{trefoil} \\
\begin{array}{c}
\text{trefoil} \\
C(\text{trefoil}) = 3
\end{array}
\end{array}
\]

**Exercise** Prove $c(\text{trefoil}) = 3$.

**Claim** If $c(K) = 1$, then $K$ is the unknot.

\[
\begin{align*}
\begin{array}{c}
a \\
b \\
c \\
d
\end{array}
\end{align*}
\]

Organize $a, b, c, d$ into all sets of two pairs:

\[
\begin{align*}
(a,b) 
& (c,d) \\
(a,c) 
& (b,d) \\
(a,d) 
& (b,c)
\end{align*}
\]

\[
\begin{array}{c}
\text{R-moves} \\
\text{R-moves} \\
\text{R-moves}
\end{array}
\]

\[
\begin{array}{c}
\text{to zero} \\
\text{to zero} \\
\text{to zero}
\end{array}
\]

\[
\begin{array}{c}
\text{one crossing}
\end{array}
\]

**Def** Let $h: \mathbb{R}^3 \to \mathbb{R}$ s.t. $h(x, y, z) = z$.

- Let $K$ be a knot in $\mathbb{R}^3$. We can move the vertices of $K$ slightly so that no edge of $K$ is parallel to the $xy$-plane.
Hence all vertices of $K$ are
local maxima, local minima or neither
\[ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \]

The bridge number of $K$ is the minimal number of local maxima over all knots equivalent to $K$. (denoted $\beta(K)$).

**Theorem** If $\beta(K) = 1$, then $K$ is an unknot.

**Idea** By Rolle's Theorem, if any plane parallel to the $xy$-plane meets $K$ in 3 or more points then $K$ has at least 2 maxima. Hence, all planes meet $K$ in 0, 1 or 2 points.

Let $b$ be the height of the first vertex of $K$ that is not a maximum.

\[ h^{-1}(b) \]

\[ h^{-1}(p) \cap K = \mathbb{E} V_1, V_2 \]
The triangle $\Delta v_1, v_2, m$ defines a $E_0D_0$ of $K$ that reduces its total height.

Repeat this process until $K$ consists of exactly 4 vertices, then appeal to the homework problem.

$Ex.$ $\beta(trefoil) = 2$

**Width**

Let $c_1 < c_2 < \ldots < c_{n+1}$ be the set of critical values of $h/K$.

Choose regular values $r_i$ s.t. $c_i < r_i < c_{i+1}$ for $1 \leq i \leq n$.

$$\sum_{i=1}^{n} |K \cap h^{-1}(r_i)|.$$

The width of a knot is the minimum value of over all knots equivalent to $K$ (denoted $w(K)$).
Theorems and open questions

Recall \( K, \# K_2 \) denotes the connected sum of \( K_1 \) and \( K_2 \)

\[
\begin{align*}
K_1 \# K_2
\end{align*}
\]

- If \( K \) cannot be decomposed as a non-trivial connected sum, we say \( K \) is prime.

\[\text{Ex}\]

\[\text{Conj}\]
\[
\mu (K, \# K_2) = \mu (K_1) + \mu (K_2)
\]

\[\text{Thm}\] (Scharlemann) 85
\[
\text{If } \mu (K, \# K_2) = 1, \text{ then } K_1 \text{ or } K_2 \text{ is the unknot.}
\]

\[\text{Pf}\]
Very Hard!

\[\text{Conj}\]
\[
c (K, \# K_2) = c (K_1) + c (K_2)
\]

\[\text{Thm}\] (Kauffman, Murasugi, Thistlethwaite) 88
\[
\text{If } K_1 \text{ and } K_2 \text{ are alternating, then } c (K, \# K_2) = c (K_1) + c (K_2)
\]

\[\text{Def}\]
A knot is alternating if the crossings alternate under, over, under as you travel along some diagram of the knot.
**Thm. (Schubert)**

\[ \beta(K_1 \# K_2) = \beta(K_1) + \beta(K_2) - 1 \]

**Pf.** Hard. See paper by Schultens.

**Ex.**

\[
\begin{align*}
\beta(K_1 \# K_2) &= \beta(K_1) + \beta(K_2) \\
\end{align*}
\]

\[
\Rightarrow \\
\beta(K_1 \# K_2) < \beta(K_1) + \beta(K_2)
\]

**Conj.**

\[ w(K_1 \# K_2) = w(K_1) + w(K_2) - 1 \]

False!

**Thm. (Blair, Tomova)** There exist knots $K_1$ and $K_2$ such that $w(K_1 \# K_2) < w(K_1) + w(K_2) - 1$. 
Outline
- Wrap-up classical invariants
- Tait conjectures
- Kauffman bracket

From last time

\(U(K)\) is the unknotting number of \(K\)
\(\beta(K)\) is the bridge number of \(K\)
\(c(K)\) is the crossing number of \(K\)
\(w(K)\) is the width of \(K\)

\[\text{Ex}\]

\[\begin{array}{c}
\text{U (trefoil)} = 1 \\
\text{\beta (trefoil)} = 2 \\
\text{c (trefoil)} = 3 \\
\text{w (trefoil)} = 8 \\
\end{array}\]

\[\text{Theorem (Scharlemann)}\]
\(U(K_1 \# K_2) = U(K_1) + U(K_2)\)
holds iff \(U(K_1 \# K_2) = 1\).

\[\text{Theorem (Schubert)}\]
\(\beta(K_1 \# K_2) = \beta(K_1) + \beta(K_2) - 1\)
\[\text{Theorem}\]
If \(\beta(K) = 1\), then \(K\) is the unknot.
Def: A diagram of a knot is alternating iff traversing the knot diagram starting at any point and recording the crossings you meet as O (for over) or U (for under) produces the pattern OUOU...OU or UOUOU...OU.

Ex: 

Def: A knot K is alternating if K has some diagram that is alternating.

Thm: If β(K) = 2, then K is alternating.

Thm (Kauffman, Murasugi, Thistlethwaite)

If K₁ and K₂ are alternating then c(K₁ # K₂) = c(K₁) + c(K₂).

Tait Conjectures (late 1800s)

1) Reduced alternating diagrams realize minimal crossing number.

2) Any two reduced alternating diagrams of a given knot have equal writhe.
3) (flyping conjecture) Any two reduced alternating diagrams of a knot have the same number of crossings.

**Def:** A diagram is **reduced** if it has no isthmuses.

**Isthmus:** There exists a circle in the plane meeting the knot diagram in a single crossing.

\[
\begin{array}{c}
\text{L} \times \text{R} \\
\end{array}
\rightarrow 
\begin{array}{c}
\text{L} = \text{B} \\
\end{array}
\]

**Def:** An **orientation** on a knot is a choice of forward direction (choice = consistent choice of unit tangent vector).

**Def:** The **writhe** of a knot diagram is the number of positive crossings minus the number of negative crossings.

\[
\begin{array}{c}
\text{positive} \\
\end{array}
\]
\[
\begin{array}{c}
\text{negative} \\
\end{array}
\]
Ex. \[ w(D) = 2 - 2 = 0 \]

\[ \rightarrow \quad \text{negative} \]

Def. A flype is the following diagramatic move

Thm. Any two reduce alternating diagrams of a knot are related by a sequence of flypes.

*flype is an old Scottish word meaning "to fold or turn back".*

All of the tait conjectures were proved by using the Jones polynomial.
Kaufman bracket polynomial

**Def.** The Kaufman bracket Polynomial of a link diagram D is a Laurent polynomial \( \langle D \rangle \in \mathbb{Z}[A^{\pm 1}] \), defined by the rules

1. It is invariant under RO moves.

\[
\langle \hat{\circ} \hat{\circ} \rangle = A \langle \hat{\circ} \rangle + A^{-1} \langle \hat{\circ} \hat{\circ} \rangle 
\]

(skein relation)

2. \( \langle D \parallel U \rangle = (-A^2-A^{-2}) \langle D \rangle \) where U is any crossing less diagram of the unknot.

3. \( \langle U \rangle = 1 \) (this is a normalization).

**Ex.**

\[
\langle \infty \rangle = A \langle 00 \rangle + A^{-1} \langle \infty \rangle \\
= A (-A^2-A^{-2}) \langle 0 \rangle + A^{-1} \langle 0 \rangle \\
= -A^3 - A^{-1} + A^{-1} = -A^3
\]

So, this is not a knot invariant.
Example | Suppose $D$ is related to $D^+$ by a single $R1^+$ move (i.e., $D^+ \xrightarrow{R1^+} D$)

$$
\langle D^+ \rangle = \langle \varphi_0 \rangle = A \langle \varphi_0 \rangle + A^{-1} \langle \varphi_1 \rangle \\
= A (-A^2 - A^{-2}) \langle \varphi_0 \rangle + A^{-1} \langle \varphi_1 \rangle \\
= (A (-A^2 - A^{-2}) + A^{-1}) \langle D \rangle \\
= (-A^3) \langle D \rangle
$$

Claim | The Kauffman bracket poly is invariant under $RII$ and $RIII$ moves.

$$
\langle \langle \varphi_0 \rangle \rangle = A \langle \langle \varphi_0 \rangle \rangle + A^{-1} \langle \langle \varphi_1 \rangle \rangle \\
= A (A \langle \langle \varphi_0 \rangle \rangle + A^{-1} \langle \langle \varphi_1 \rangle \rangle) \\
+ A^{-1} \left( A \langle \langle \varphi_0 \rangle \rangle + A^{-1} \langle \langle \varphi_1 \rangle \rangle \right) \\
= A^2 \langle \langle \varphi_0 \rangle \rangle + \langle \langle \varphi_1 \varphi_1 \rangle \rangle \\
- A^2 \langle \langle \varphi_0 \rangle \rangle - A^{-2} \langle \langle \varphi_1 \rangle \rangle + A^{-2} \langle \langle \varphi_1 \rangle \rangle \\
= \langle \langle \varphi_0 \rangle \rangle \checkmark
$$
Exercise: Show the Kauffman bracket poly is invariant under the $R_{III}$ move.

Correcting via writhe:

Lemma: The writhe of an oriented knot diagram is invariant under $R_{II}$ and $R_{III}$, but changes under $RI$.

pf