Outline

- mod $p$ labels
- the linear algebra of labels.

**Def** A knot diagram can be labeled mod $p$ if each edge can be labeled with an integer from 0 to $p-1$ s.t. at each crossing $2x-y-z \equiv 0 \mod p$ where $x$ is the label on the overcrossing and $y$ and $z$ are the two other labels incident to the crossing, i.e.

```
   /y
  / \
 /   \\
```

**Note:** Under this definition every knot has $p$ distinct mod $p$ labels.

**Def** A knot diagram has a non trivial labeling mod $p$ if the diagram can be labeled mod $p$ s.t. at least two distinct labels are used.
Claim: The following Reidemeister move induces a unique $\mathbb{Z}$ mod $p$ labeling on the diagram.

Must verify that $Z(Zz - x) - (Zz - y) \equiv ZZ - (Zx - Y) \mod p$.

$$Z(Zz - x) - (Zz - y) = 4Z - 2x - 2z + y$$
$$= ZZ - (Zx - Y). \checkmark$$

Hence, a labeling of the diagram before this Reidemeister move induces a unique labeling of the diagram after the move.

Exercise | Verify this for all remaining Reidemeister moves.

Theorem | The number of mod $p$ labels of any diagram is a knot invariant.
Example

Fig. 8 knot has a non-trivial labeling mod 5.

\[ \begin{align*}
2b - a - x &\equiv 0 \mod 5 \\
2a - b - x &\equiv 0 \mod 5 \\
2x - b - a &\equiv 0 \mod 5
\end{align*} \]

\[ \begin{align*}
1) & x = 2b - a \\
2a - b &\equiv 2b - a \\
3a &\equiv 3b \\
\alpha &\equiv b \quad \text{working mod 5}
\end{align*} \]

So, the trefoil has only trivial labelings mod 5.

Hence, mod 5 labels distinguish the figure 8 knot and the trefoil.

To show that the number of mod p labelings is a knot invariant we must show that any labeling of a knot before any Reidemeister move induces a unique labeling after the Reidemeister move.
Recall that the integers mod $p$ where $p$ is prime is a field where as the integers mod $p$ where $p$ is not prime is a ring.

We can use linear algebra to calculate the number of $p$-colorings of mod $p$ labelings of a knot diagram.

(for simplicity, we will assume $p$ is a prime so that we are working with vector spaces instead of modules)

Step 1: Assign to each arc of the diagram a variable $X_i$.

Step 2: Each crossing induces a linear equation over the integers mod $p$.

1) $2X_1 - X_2 - X_3 \equiv 0$
2) $2X_3 - X_1 - X_4 \equiv 0$
3) $2X_4 - X_2 - X_1 \equiv 0$
4) $2X_2 - X_3 - X_4 \equiv 0$
Step 3: We produce a matrix equation mod $p$.

$$
\begin{bmatrix}
2 & -1 & -1 & 0 \\
-1 & 0 & 2 & -1 \\
-1 & -1 & 0 & 2 \\
0 & 2 & -1 & -1
\end{bmatrix} \begin{bmatrix}
\begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{array}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
$$

$A\vec{x} = \vec{0}$

Step 4: From linear algebra over finite fields, the number of solutions to this matrix eq. mod $p$ is $p^{(\text{nullity of } A)}$. (Recall nullity of $A$ is the dimension of the null space of $A$).

Step 5: Find the nullity of $A$ using row reduction mod $p$.

$$
\begin{bmatrix}
2 & -1 & -1 & 0 \\
-1 & 0 & 2 & -1 \\
-1 & -1 & 0 & 2 \\
0 & 2 & -1 & -1
\end{bmatrix} \xrightarrow{R_1 + R_2 + R_3 \rightarrow R_1} \begin{bmatrix}
0 & -1 & 2 & -1 \\
-1 & 0 & 2 & -1 \\
-1 & -1 & 0 & 2 \\
0 & 2 & -1 & -1
\end{bmatrix} \xrightarrow{R_3 - R_2 \rightarrow R_3} \begin{bmatrix}
0 & 1 & 3 & -1 \\
-1 & 0 & 2 & -1 \\
0 & -1 & 2 & 3 \\
0 & 2 & -1 & -1
\end{bmatrix} \xrightarrow{R_4 + 2R_1 \rightarrow R_4} \begin{bmatrix}
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$
\[
\begin{bmatrix}
2 & 4 & 4 & 0 \\
4 & 0 & 2 & 4 \\
4 & 4 & 0 & 2 \\
0 & 2 & 4 & 4
\end{bmatrix} \rightarrow 
\begin{bmatrix}
2 & 4 & 4 & 0 \\
0 & 2 & 4 & 4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \rightarrow 
\begin{bmatrix}
2 & 4 & 4 & 0 \\
0 & 1 & 2 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

so, nullity of \( A \) is 2.

Hence, the figure 8 knot has \( S^2 \mod 5 \) labels.