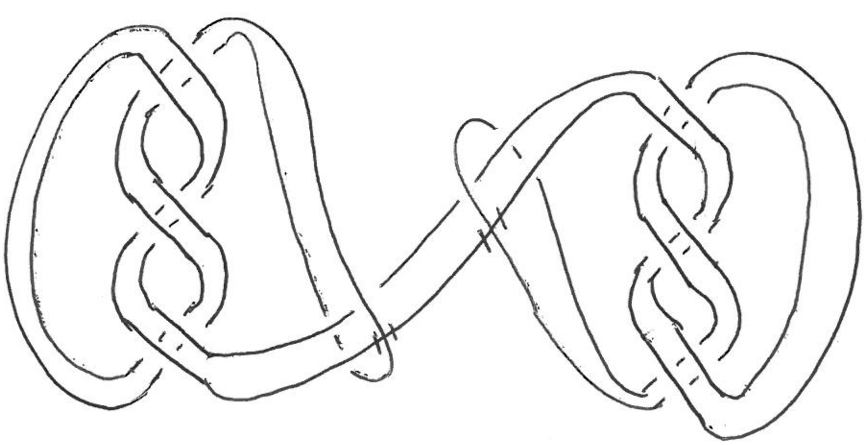
Knot Theory Day 5

Outline

- Monster Unknots
- Colorability
-

Thm (Hass and Lagarias)
Any unknotted knot diagram D with n crossings can be transformed to the trivial knot diagram using at most $2^{(10^{11})n}$ Reidermeister moves.

<u>Ex</u> "Cousin it"

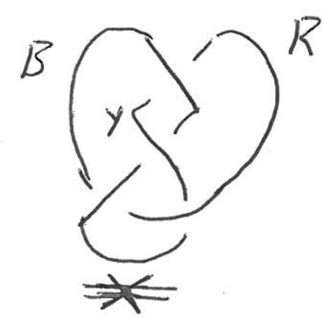


Colorability of knots (Due to Ralf Fox)

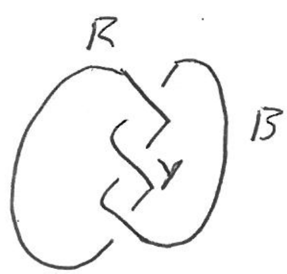
<u>Def</u> A knot diagram is <u>colorable</u> if each arc can be drawn using one of three collors (red, yellow, blue) s.t.

1) At least two colors are used
2) at any crossing where two colors ~~are~~ appear, all three appear

<u>Ex</u> Not colorable



Colorable



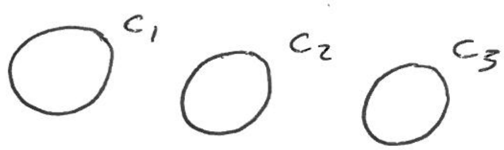
By the theorem of Alexander and Briggs, Collorability is a <u>knot</u> <u>invariant</u> if it is preserved under each of the Reidermeister moves.

For H.W. Show Colorability is preserved under R III

Hence, colorability is an <u>invariant</u> of knots.
(i.e. if K is equivalent to J and J is colorable (not colorable) then K is colorable (not colorable).

<u>Note</u> The number of 3-colorings is also an invariant of knots, denoted $c_3(K)$.

<u>Ex</u>



$c_3(\text{unknot}) = 3$

$c_3(\text{trefoil}) = 9$

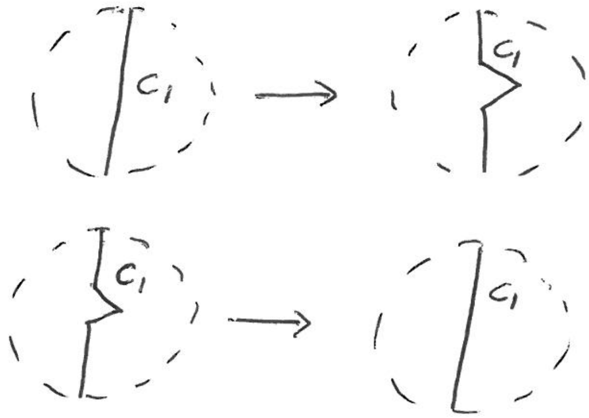
<u>Ex</u> $c_3(K_1 \amalg K_2) = c_3(K_1) \cdot c_3(K_2)$

<u>Pf</u> Since c_3 is an invariant we can choose to color the diagram where $P(K_1) \cap P(K_2) = \emptyset$.

For every fix coloring of K_1 we have $c_3(K_2)$ colorings of $K_1 \amalg K_2$.
Hence $c_3(K_1 \amalg K_2) = c_3(K_1) \cdot c_3(K_2)$

Claim 1: Colorability is preserved under R0

Let c_1 be some color



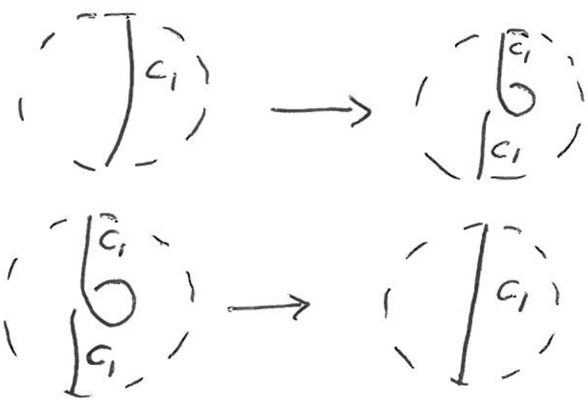
neither of these moves changes

① The fact that at least two colors are used

② The fact that at any crossing where 2 colors are used, all 3 are used.

Claim 2: Colorability is preserved under R I

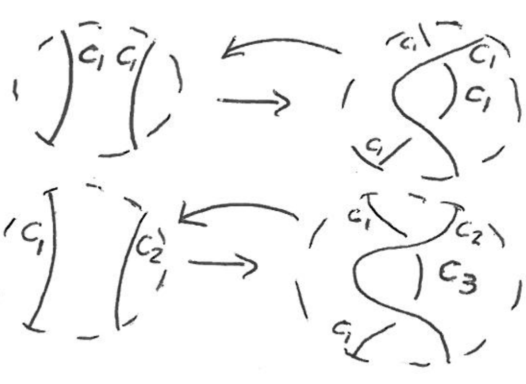
Let c_1 be some color.



neither of these moves change ① or ②

Claim 3: Colorability is preserved under R II

Let c_1 be some color,
Let c_2 be some distinct color.



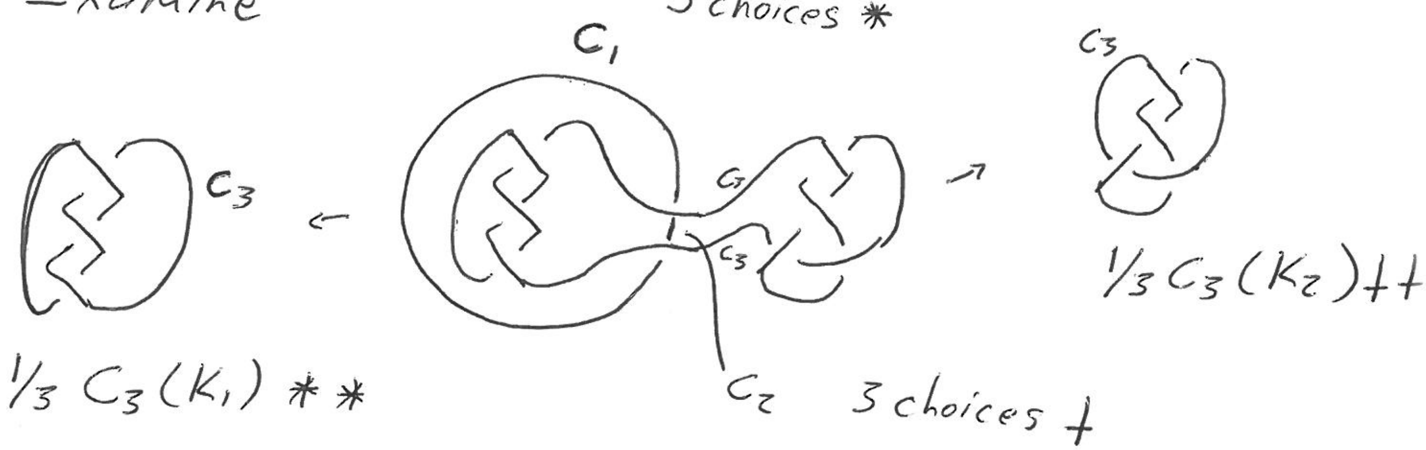
Under all possible initial colorings neither of these moves changes ① or ②

Ex| Show $C_3(K_1 \# K_2) = \frac{1}{3} C_3(K_1) \cdot C_5(K_2)$.

Pf|

$$C_3(\text{unknot} \sqcup K_1 \# K_2) = 3 \cdot C_3(K_1 \# K_2)$$

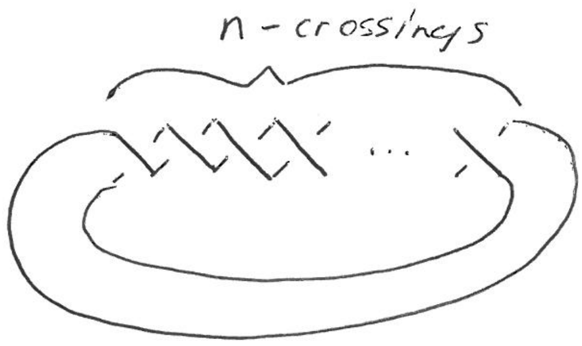
Examine



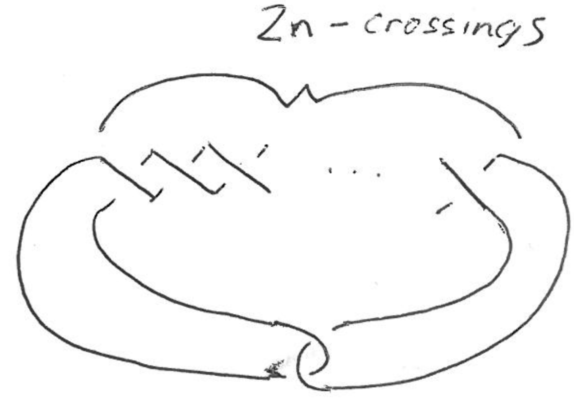
$$C_3(\text{unknot} \sqcup K_1 \# K_2) = \overset{*}{(3)}\overset{+}{(3)}\overset{**}{\left(\tfrac{1}{3} C_3(K_1)\right)} \cdot \underset{++}{\left(\tfrac{1}{3} C_3(K_2)\right)}$$

So, $C_3(K_1 \# K_2) = \frac{1}{3} C_3(K_1) \cdot C_3(K_2)$.

Some knots in the H.W.



(2, n) torus knot



n-twisted double of the unknot.

mod p lables

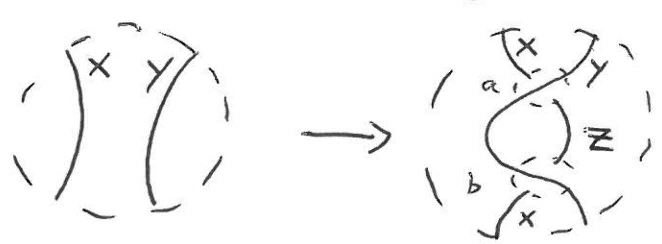
Def) A knot diagram can be labeled mod p if each edge can be labled with an integer from 0 to p-1 s.t.

1) At each crossing $2x-y-z \equiv 0 \bmod p$ where x is the lable on the overcrossing and y and z are the ~~lables on the~~ other two lables

2) At least two distinct lables are used.

Thm) If some diagram of a knot can be labled mod p then every diagram can be labled mod p.

Example) Ability to be labled mod p is preserved by the following Reidermeister move



Choose lables x and y for the strands in the left fig.

Show that there exists a value z s.t. criteria 1) of the def. of labled mop p holds at each crossing in the right fig.

$$\left.\begin{array}{l} \text{From crossing } a: \quad 2y-x-z \equiv 0 \bmod p \\ \text{From crossing } b: \quad 2y-x-z \equiv 0 \bmod p \end{array}\right\} z \equiv 2y-x \bmod p$$

$\square$