Outline
- Definition of knot
- Definition of knot equivalence
- The issue of rotations

Definition of knot

Let $p$ and $q$ be distinct points in $\mathbb{R}^3$. Let $[p, q]$ denote the line segment from $p$ to $q$.

Let $(p_1, p_2, \ldots, p_n)$ be an ordered set of distinct points in $\mathbb{R}^3$. $[p_1, p_2] \cup [p_2, p_3] \cup \ldots \cup [p_{n-1}, p_n] \cup [p_n, p_1]$ is a simple closed polygonal curve if each segment intersects the union of all other segments in exactly its two end points.

Definition: A knot is a simple closed polygonal curve in $\mathbb{R}^3$.

Example:

```
   (0,1,0)                (0,1,0)
   (0,0,0)                (-1,0,0)
   (1,0,0)                (0,0,0)
```

"Different" from
Note: When I draw a smooth knot, I am thinking of it being the union of many line segments. (In fact, smooth and polygonal knot theory are equivalent)

The following helps us eliminate some of the ambiguity in our definition of knot

Def: If the ordered set \((P_1, P_2, \ldots, P_n)\) defines a knot and no proper ordered sub set defines the same knot, \(P_1, \ldots, P_n\) are called the vertices of the knot.

Def: A link is the finite union of disjoint knots.

Equivalence of knots

Def: Given a knot \(K\) determined by \((P_1, P_2, \ldots, P_n)\), a knot \(L\) determined by \((P_0, P_1, P_2, \ldots, P_n)\) is an elementary deformation of \(K\) if

1. \(P_0\) is not collinear with \(P_1\) and \(P_n\)
2. The triangle spanned by \((P_0, P_1, P_n)\) intersects \(K\) only in \([P_1, P_n]\).
Equivalence of knots

Def: \( J \) is an elementary deformation of \( K \) if one of the knots \( J \) and \( K \) is determined by \((p_1, p_2, \ldots, p_n)\) and the other is determined by \((p_0, p_1, \ldots, p_n)\) s.t.

1. \( p_0 \) is not collinear with \( p_1 \) and \( p_n \)
2. The triangle spanned by \((p_0, p_1, p_n)\) intersects \([p_1, p_2, \ldots, v[p_n, p_1]]\) only in \([p_1, p_n]\).
Def: Knots $K$ and $J$ are equivalent if there is a sequence of knots $K = K_0, K_1, K_2, \ldots, K_n = J$ so that $K_{i+1}$ is an elementary deformation of $K_i$ for $0 \leq i \leq n-1$.

Claim: This notion of equivalence defines an equivalence relation on the set of all knots.

Recall that in defines an equivalence relation on a set $X$ if

1. $a \equiv a$ (reflexivity)
2. $a \equiv b \implies b \equiv a$ (symmetry)
3. $a \equiv b$ and $b \equiv c \implies a \equiv c$ (transitivity)
Proof of claim

1. \( K \) is equivalent to \( K \) via the sequence \( K \).

2. If \( K \) is equivalent to \( J \) via the sequence 
   \[ K = K_0, K_1, K_2, \ldots, K_n = J, \]
   then \( J \) is equivalent to \( K \) via the sequence 
   \[ J = K_n, K_{n-1}, \ldots, K_0 = K. \]

3. If \( K \) is equivalent to \( J \) via 
   \[ K = K_0, \ldots, K_n = J \]
   and 
   \( J \) is equivalent to \( L \) via 
   \[ J = K_{n+1}, \ldots, K_{m+n} = L, \]
   then 
   \( K \) is equivalent to \( L \) via 
   \[ K = K_0, K_1, \ldots, K_{m+n} = L. \]

A key lemma

Lemma: Let \( K \) be a knot determined by the points 
\( (p_1, \ldots, p_n) \). Show that there exists \( \varepsilon \) s.t. if 
\( d(p_1, p_1') < \varepsilon \), then \( K \) is equivalent to the knot 
determined by \( (p_1', p_2, \ldots, p_n) \).
Proof: By def. of simple closed polygonal curve,
$[P_n, P_1]$ is disjoint from $[P_2, P_3] \cup \ldots \cup [P_{n-2}, P_{n-1}]$ and $[P_1, P_2]$ is disjoint from $[P_3, P_4] \cup \ldots \cup [P_{n-1}, P_n]$.
Since all of these spaces are compact,
$\exists \varepsilon_1$ s.t. $d(x, y) > \varepsilon_1$ for any $x \in [P_n, P_1]$ and any $y \in [P_2, P_3] \cup \ldots \cup [P_{n-2}, P_{n-1}]$.

$\exists \varepsilon_2$ s.t. $d(x, y) > \varepsilon_2$ for any $x \in [P_1, P_2]$ and any $y \in [P_3, P_4] \cup \ldots \cup [P_{n-1}, P_n]$.
Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$.
Let $p_1'$ be any point that is not collinear to $p_n$ and $p_1$ and not collinear to $p_1$ and $p_2$ s.t. $d(p_1', p_1) < \min(\varepsilon_1, \varepsilon_2)$.
Let $\mathcal{E}(E, \varepsilon)$ denote the collection of all points in $\mathbb{R}^3$ that are within $\varepsilon$ of some point on the interval $[P_1, P_2]$.

Since the triangle spanned by $(P_n, P_1, P_1')$ is contained in $\mathcal{E}(E, [P_n, P_1])$, then there is an elementary deformation from the knot defined by $(P_1, P_2, \ldots, P_n)$ to the knot defined by $(P_1', P_1, P_2, \ldots, P_n)$. Since the triangle spanned by $(P_1', P_1, P_2)$ is contained in $\mathcal{E}(E, [P_1, P_2])$, there is an elementary deformation from $(P_1', P_1, P_2, \ldots, P_n)$ to $(P_1', P_2, \ldots, P_n)$. 
If \( P_1 \) is collinear to \( P_n \) and \( P_1 \), or is collinear to \( P_1 \) and \( P_2 \), a similar elementary deformation can be found. Thus, \( K \) is equivalent to \( (P_1', P_2, \ldots, P_n) \). \( \Box \)

(HW) Th. Given a knot \( K \), there is a positive constant \( E_K \) s.t. every vertex of \( K \) can be moved a distance of at most \( E_K \) without changing the equivalence class of the knot.

The previous theorem can be used to show that 2 knots that differ by a translation or rotation are equivalent.
Outline
- Knot Projections
- Knot diagrams
- Orientations on knots
- Reidemeister Moves
- Colorability.

Knot Projections
Let \( P : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) via \( P(x, y, z) = (x, y) \).

A projection of a knot \( K \) is the image of \( P|_K \) (\( P \) restricted to \( K \)).

Example

```
projection of a fig. 8 knot.
```

Definition
A knot projection is regular if no three points on the knot project to the same point and if two points project to the same point, neither is a vertex.

(i.e. we don't see)

```
X  or  \n```

A knot diagram is a regular knot projection together with information at each crossing that dictates which strand passes over and which passes under.

Examples

\[ \text{\includegraphics{example1.png}} \quad \text{or} \quad \text{\includegraphics{example2.png}} \]

Question: If two knots have identical knot diagrams, are they the same knot?

Theorem 1: Let K be a knot determined by \((p_1, \ldots, p_n)\).

\[ \forall t > 0 \ \exists K' \text{ determined by } (q_1, \ldots, q_n) \ s.t. \text{ the distance from } q_i \text{ to } p_i \text{ is less than } t \text{ and } K' \text{ has a regular projection.} \]

Corollary: Every knot is equivalent to a knot with a regular projection.

Idea:

\[ \begin{align*}
\text{\includegraphics{idea1.png}} & \quad \Rightarrow \quad \text{\includegraphics{idea2.png}} \\
\text{\includegraphics{idea3.png}} & \quad \Rightarrow \quad \text{\includegraphics{idea4.png}}
\end{align*} \]
Theorem 2: Let $K$ be a knot determined by $(p_1, \ldots, p_n)$ s.t. $K$ has a regular projection. There exists a constant $t > 0$ s.t. if $K'$ is determined by $(q_1, \ldots, q_n)$ with each $q_i$ within distance $t$ of $p_i$, then $K'$ has a regular projection.

"Regular projections are stable"

Idea:

\[ \text{e-perturbation} \]

Theorem 3: If $J$ and $K$ have identical diagrams, then they are equivalent.

Proof: Let $K$ be determined by $(p_1, \ldots, p_n)$ and let $J$ be determined by $(q_1, \ldots, q_m)$. Assume that no point $p_1, \ldots, p_n$ and no point $q_1, \ldots, q_m$ project to a crossing of the diagram.

Claim: We can add vertices to $K$ and $J$ s.t.

\[ P(p_1) = P(q_1), \ldots, P(p_n) = P(q_m). \]
Next, we construct a sequence of elementary deformations taking $K$ to $J$.

Suppose $P([P_i, P_{i+1}])$ is disjoint from all crossings.

---

In this case we have the option to use e.d. to bring the $q_i$ up or bring the $P_i$ down.

What about the vertices near a crossing?

---

In this case, bring the $q_i$ up to prevent from constructing an invalid elementary deformation.

Since $\{P_1, \ldots, P_n\} = \{q_1, \ldots, q_n\}$ in $\mathbb{R}^3$, then $K$ is equivalent to $J$.\[\]
Knot Theory Day 4

Outline
- Knot diagrams
- Reidemeister Moves
- Colorability

From last time

**Theorem** If $J$ and $K$ have identical diagrams, then they are equivalent.

Recall $P : \mathbb{R}^3 \to \mathbb{R}^2$ via $P(x, y, z) = (x, y)$

**Pf** Last time we showed that we can assume $K$ is defined by $(p_1, \ldots, p_n)$ and $J$ is defined by $(q_1, \ldots, q_n)$ so that

$P(p_i) = P(q_i)$ for each $1 \leq i \leq n$.

Additionally, we can assume $P([P_i, P_{i+1}] \cup [P_n, P_1])$ intersects at most one crossing.
Claim | We can find a sequence of elementary deformations of $K$ and $J$ s.t. $P_i = Q_i, \ldots, P_n = Q_n$ (if we can accomplish this, then we are done).

Case 1 | Suppose $P([P_{i-1}, P_i] \cup [P_i, P_{i+1}])$ is disjoint from all crossings. Then there is a sequence of two elementary deformations taking $P_i$ to $Q_i$ and two taking $Q_i$ to $P_i$.

Examine $P^{-1}([P_{i-1}, P_i]) \cup P^{-1}([P_i, P_{i+1}])$, the union of two vertical, infinite bonds.

- Move $P_i$ to $Q_i$ while fixing $P_j$ and $Q_j$ for $i \neq j$.
- First, preform an e.d. for $K$ along $\triangle P_{i-1}, Q_i, P_i$ to form $K'$.
- Second, preform an e.d. for $K'$ along $\triangle P_i, Q_i, P_{i+1}$.
Case 2 | Suppose $P([P_{i-1}, P_i] \cup [P_i, P_{i+1}])$ meets exactly one crossing. Then there is a sequence of two elementary deformations taking $q_i$ to $q_e$ or $q_e$ to $q_i$ (but not both).

Examine $P^{-1}([P_{i-1}, P_i]) \cup P^{-1}([P_i, P_{i+1}])$ and assume we see a under crossing on $P^{-1}([P_{i-1}, P_i])$. (i.e. $\times$)

Move $q_i$ to $P_i$.

1. Apply an e.d. to $J$ along $\Delta q_i, P_i, q_{i-1}$
2. Apply an e.d. along $\Delta q_i, P_i, q_{i+1}$

Hence, we found a sequence of e.d. taking $K$ to $J$. $\Box$
Reidemeister moves for knot diagrams

\[ R_0 \quad \xrightarrow{\text{ }} \quad R_1 \quad \xrightarrow{\text{ this is the projection of an elementary deformation }} \]

\[ R_{II} \quad \xrightarrow{\text{ }} \quad R_{III} \]

**Thm (Alexander & Briggs)**

If two knots are equivalent, their diagrams are related by a sequence of Reidemeister moves.

**Ex**

\[ R_{II} \quad \xrightarrow{\text{ }} \quad R_1 \quad \xrightarrow{\text{ }} \quad \]
"Monster unknots"

"Cousin it"

Thm (Hass and Lagarias)
Any unknotted knot diagram $D$ with $n$ crossings can be transformed to the trivial knot diagram using at most $2^{10n}$ Reidemeister moves.
Colorability of knots (Due to Ralph Fox)

Def: A knot diagram is **colorable** if each arc can be drawn using one of three colors (red, yellow, blue) s.t.
1) At least two colors are used
2) at any crossing where two colors appear, all three appear

Ex: Not colorable

```
\[ \includegraphics[width=0.5\textwidth]{example} \]
```

Colorable

```
\[ \includegraphics[width=0.5\textwidth]{example} \]
```

By the theorem of Alexander and Briggs, colorability is a **knot invariant** if it is preserved under each of the Reidemeister moves.
Claim 1: Colorability is preserved under $R_0$.
Clearly true.

Claim 2: Colorability is preserved under $R_I$

\[
\begin{align*}
\begin{array}{c}
\text{R} \\
\text{R}
\end{array}
& \rightarrow \\
\begin{array}{c}
\text{R} \\
\text{R}
\end{array}
\end{align*}
\]

Since at least 2 colors are used and since the $R_I$ move preserves the colors disjoint from a nbh of the $R_I$ move, then the $R_I$ move preserves colorability.