Knot Theory Day 2

LAS 169

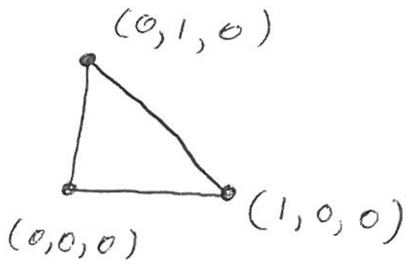
Outline

- Definition of knot
- Definition of knot equivalence
- The issue of rotations

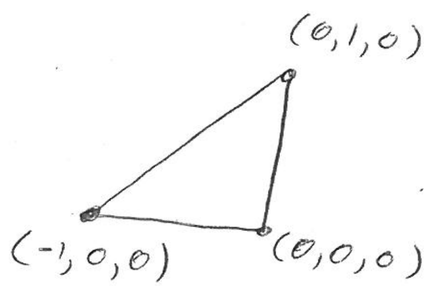
Definition of knot

- Let p and q be distinct ~~knots~~ points in $\mathbb{R}^3$. Let $[p,q]$ denote the line segment from p to q.
- Let $(p_1, p_2, \cdots, p_n)$ be an ordered set of distinct points in $\mathbb{R}^3$. $[p_1,p_2] \cup [p_2,p_3] \cup \cdots \cup [p_{n-1},p_n] \cup [p_n,p_1]$ is a simple closed polygonal curve if each segment intersects the union of all other segments in exactly its two end points.
- Def: A knot is a simple closed polygonal curve in $\mathbb{R}^3$.

Ex1



"different" from



Note: When I draw a smooth knot, I am thinking of it being the union of many line segments. (In fact, smooth and polygonal knot theory are equivalent)

The following helps us eliminate some of the ambiguity in our definition of knot

Def: If the ordered set $(p_1, p_2, \dots, p_n)$ defines a knot and no proper ordered sub set defines the same knot, $p_1, \dots, p_n$ are called the <u>vertices</u> of the knot.

Def: A link is the finite union of disjiont knots.

<u>Equivalence of knots</u>

Def: Given a knot K determined by $(p_1, p_2, \dots, p_n)$ a knot J determined by $(p_0, p_1, p_2, \dots, p_n)$ is an <u>elementary deformation</u> of K if

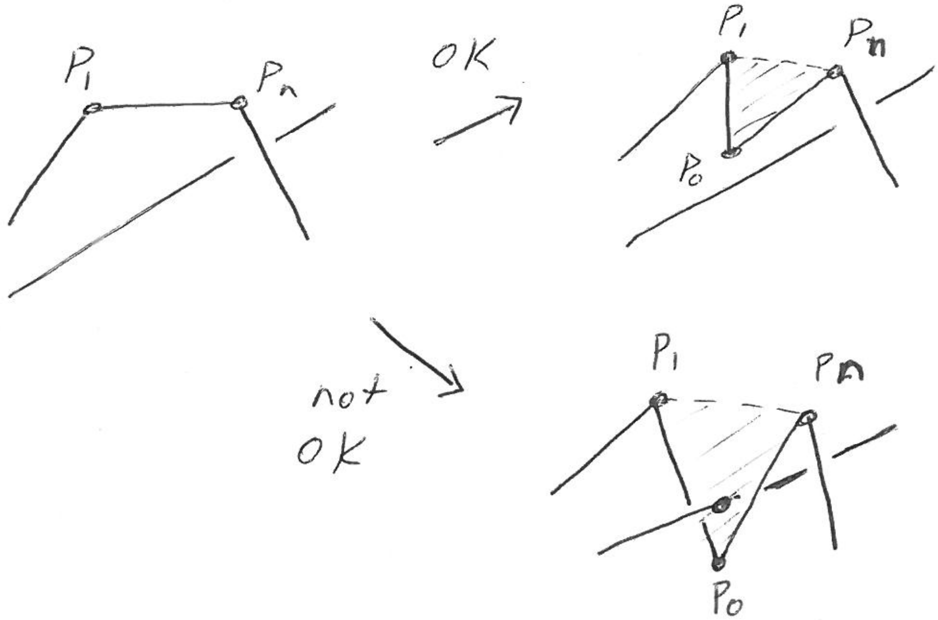
1. p_0 is not collinear with p_1 and p_n
2. The triangle spanned by (p_0, p_1, p_n) intersects K only in $[p_1, p_n]$.

Equivalence of knots

Def: J is an <u>elementary</u> <u>deformation</u> of K if one of the knots J and K is determined by $(p_1, p_2, \cdots, p_n)$ and the other is determined by $(p_0, p_1, \cdots, p_n)$ s.t.

① p_0 is not collinear with p_1 and p_n

② The triangle spanned by (p_0, p_1, p_n) intersects $[p_1, p_2] \cup \cdots \cup [p_{n-1}, p_n] \cup [p_n, p_1]$ only in $[p_1, p_n]$.



Def: Knots K and J are <u>equivalent</u> if there is a sequence of knots $K = K_0, K_1, K_2, \ldots, K_n = J$ s.t. K_{i+1} is an elementary deformation of K_i for $0 \le i \le n-1$.

Claim: This notion of equivalence defines an equivalence relation on the set of all knots

Recall that $\sim$ defines an equivalence relation on a set X if
1. $a \sim a$ (reflexivity)
2. $a \sim b \Rightarrow b \sim a$ (symetry)
3. $a \sim b$ and $b \sim c \Rightarrow a \sim c$ (transitivity)

Proof of claim

① K is equivalent to K via the sequence K.

② If K is equivalent to J via the sequence $K = K_0, K_1, K_2, \ldots, K_n = J$, then J is equivalent to K via the sequence $J = K_n, K_{n-1}, \ldots, K_0 = K$.

③ If K is equvalent to J via $K = K_0, \ldots, K_n = J$ and J is equivalent to L via $J = K_n, \ldots, K_{m+n} = L$, then K is equivalent to L via $K = K_0, K_1, \ldots, K_{m+n} = L$.

A key lemma

Lemma: Let K be a knot determined by the points $(P_1, \ldots, P_n)$. Show that there exists ε s.t. if $d(P_1, P_1') < \varepsilon$, then K is equivalent to the knot determined by $(P_1', P_2, \ldots, P_n)$.

Proof/ By def. of simple closed polygonal curve,

$[P_n, P_1]$ is disjiont from $[P_2, P_3] \cup \cdots \cup [P_{n-2}, P_{n-1}]$ and

$[P_1, P_2]$ is disjiont from $[P_3, P_4] \cup \cdots \cup [P_{n-1}, P_n]$.

Since all of these spaces are compact,

$\exists\, \varepsilon_1$ s.t. $d(x, y) > \varepsilon_1$ for any $x \in [P_n, P_1]$ and any $y \in [P_2, P_3] \cup \cdots \cup [P_{n-2}, P_{n-1}]$

$\exists\, \varepsilon_2$ s.t. $d(x, y) > \varepsilon_2$ for any $x \in [P_1, P_2]$ and any $y \in [P_3, P_4] \cup \cdots \cup [P_{n-1}, P_n]$.

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$.

Let P_1' be any point that is not colinear to P_n and P_1 and not collinear to P_1 and P_2 s.t. $d(P_1', P_1) < \min(\varepsilon_1, \varepsilon_2)$.

Let $\eta_\varepsilon([p, q])$ denote the collection of all points in $\mathbb{R}^3$ that are with in ε of some point on the interval $[p, q]$.

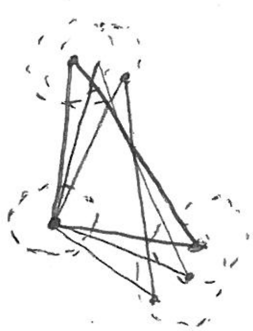
Since the triangle spanned by (P_n, P_1, P_1') is contained in $\eta_{\varepsilon_1}([P_n, P_1])$, then there is an elementary deformation from the knot defined by $(P_1, P_2, \ldots, P_n)$ to the knot difined by $(P_1', P_1, P_2, \ldots, P_n)$. Since the triangle spanned by (P_1', P_1, P_2) is contained in $\eta_{\varepsilon_2}([P_1, P_2])$, there is an elementary deformation from $(P_1', P_1, P_2, \ldots, P_n)$ to $(P_1', P_2, \ldots, P_n)$.

If P_1' is collinear to P_n and P_1 or is colinear to P_1 and P_2, a similar elementrary deformation can be found.

Thus, K is equivalent to $(P_1', P_2, \dots, P_n)$. $\square$

(HW) Thm Given a knot K, there is a positive constant ε_K s.t. every vertex of K can be moved a distance of at most ε_K without changing the equivalence class of the knot.

The previous theorem can be used to show that 2 knots that differ by a translation or a ~~reflection~~ rotation are equivalent



Outline

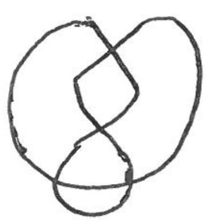
- Knot Projections
- Knot diagrams
- ~~Orientations on knots~~
- Reidermeister Moves
- Colorability.

Knot Projections

Let $P: \mathbb{R}^3 \to \mathbb{R}^2$ via $P(x,y,z) = (x,y)$.

A <u>projection</u> of a knot K is the image of $P|_K$ (P restricted to K).

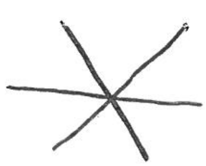
<u>Ex</u>



projection of a fig. 8 knot.

<u>Def</u> A knot projection is <u>regular</u> if no three points on the knot project to the same point and if two points project to the same point, neither is a vertex.

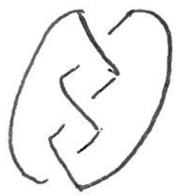
(i.e. we don't see)



or

A <u>knot diagram</u> is a regular knot projection together with information at each crossing that dictates which strand passes over and which passes under.

Examples

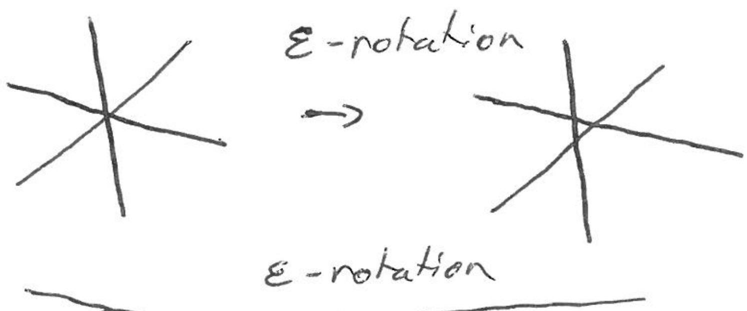
 or 

Question: If two knots have identical knot diagrams, are they the same knot?

<u>Theorem 1</u>: Let K be a knot determined by $(p_1, \dots, p_n)$. $\forall t > 0\ \exists K'$ determined by $(q_1, \dots, q_n)$ s.t. the distance from q_i to p_i is less than t and K' has a regular projection.

<u>Cor</u> Every knot is equivalent to a knot with a regular projection.

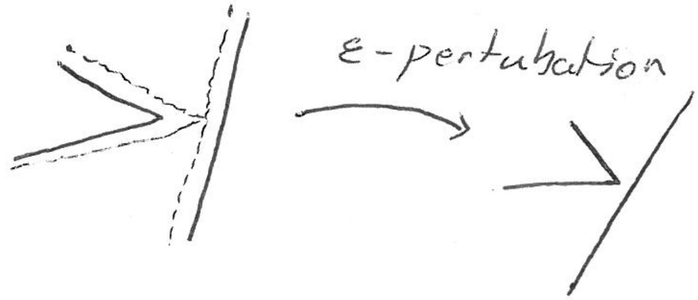
Idea:



Theorem 2 Let K be a knot determined by $(p_1, \dots, p_n)$ s.t. K has a regular projection. There exists a constant $t > 0$ s.t. if K' is determined by $(q_1, \dots, q_n)$ with each q_i within distanct t of p_i, then K' has a regular projection.

"Regular projections are stable"

Idea



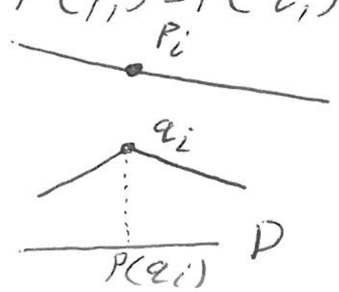
Theorem 3 If J and K have identical diagrams, then they are equivalent.

Pf Let K be determined by $(p_1, \dots, p_n)$ and let J be determined by $(q_1, \dots, q_m)$.

Assume that no point $p_1, \dots, p_n$ and no point $q_1, \dots, q_m$ project to a crossing of the diagram.

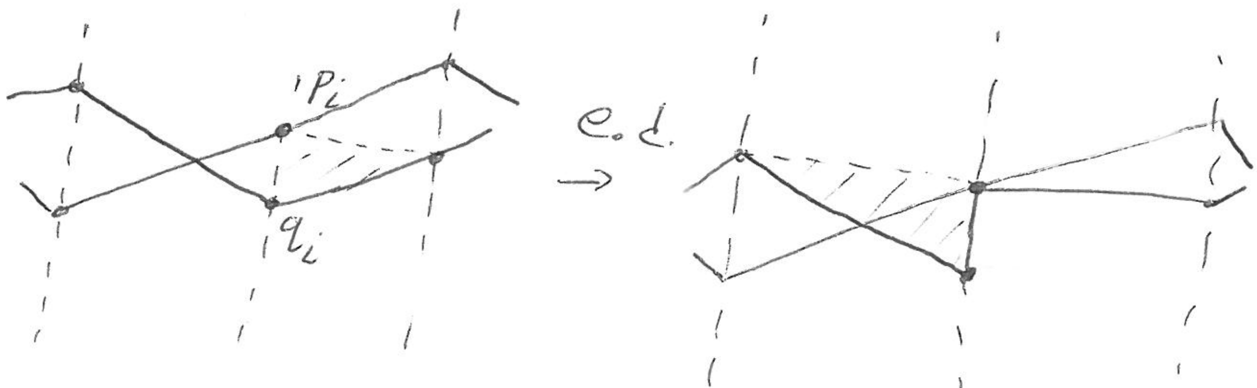
Claim: We can add vertices to K and J s.t.

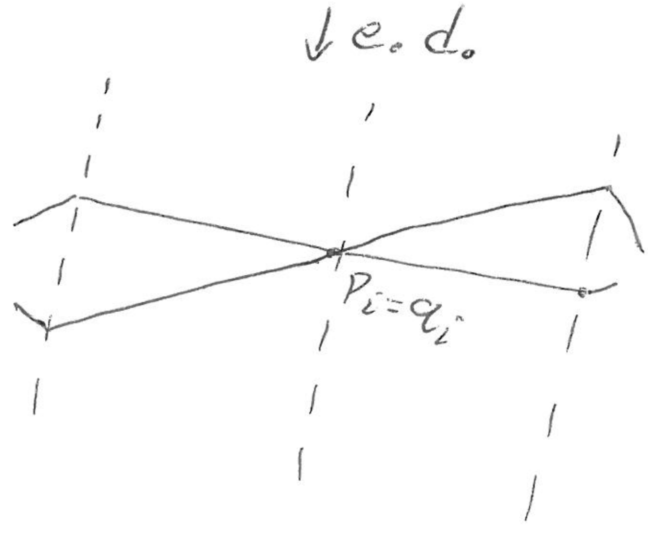
$$P(p_1) = P(q_1), \dots, P(p_n) = P(q_n).$$



Next, we construct a sequence of elementary deformations taking K to J.

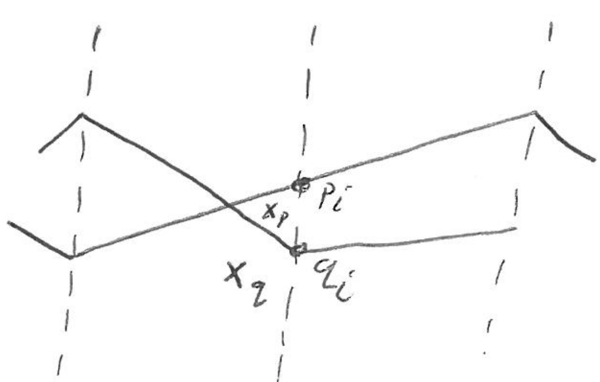
Suppose $P\left(\begin{matrix}[p_{i-1}, p_i] \cup \\ [p_i, p_{i+1}]\end{matrix}\right)$ is disjiont from all crossings.





- In this case we have the option to use e.d. to bring the q_i up or bring the p_i down.

What about the vertices near a crossing?



← in this case, bring the q_i up, to prevent from constructing an invalad elementary deformation

Since $\{p_1, \ldots, p_n\} = \{q_1, \ldots, q_n\}$ in $\mathbb{R}^3$, then K is equivalent to J. $\square$

Knot Theory Day 4

Outline
- Knot diagrams
- Reidermeister Moves
- Colorability

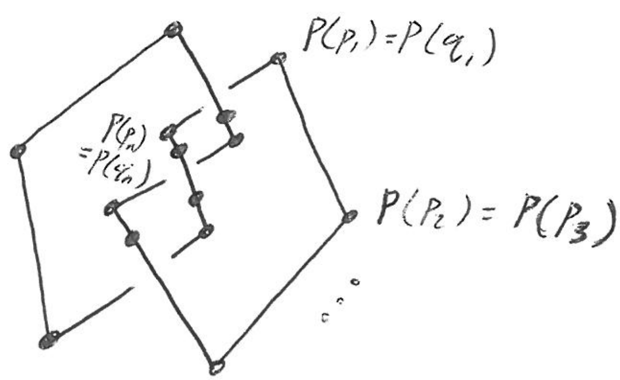
From last time

Theorem If J and K have identical diagrams, then they are equivalent.

Recall $P: \mathbb{R}^3 \to \mathbb{R}^2$ via $P(x,y,z) = (x,y)$

Pf Last time we showed that we can assume K is defined by $(p_1, \ldots, p_n)$ and J is defined by $(q_1, \ldots, q_n)$ s.t. $P(p_i) = P(q_i)$ for each $1 \le i \le n$.

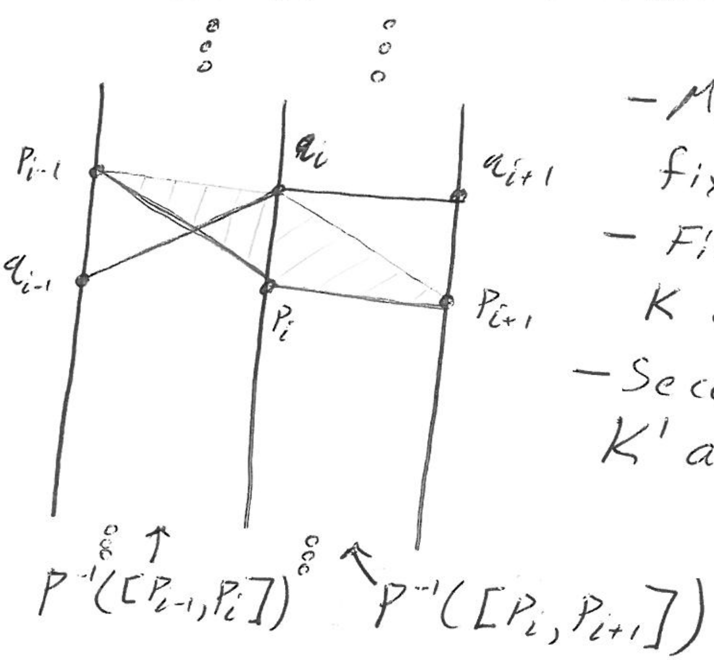
Additionally, we can assume $P([p_i, p_{i+1}] \cup [p_{i+1}, p_{i+2}])$ intersects at most one crossing.



<u>Claim</u> We can find a sequence of elementary deformations of K and J s.t. $p_1 = q_1, \ldots, p_n = q_n$ (if we can acomplish this, then we are done)

<u>Case 1</u> Suppose $P([p_{i-1}, p_i] \cup [p_i, p_{i+1}])$ is disjiont from all crossings. Then there is a sequence of two elementary deformations taking p_i to q_i (~~or~~ and two taking q_i to p_i).

Examine $P^{-1}([p_{i-1}, p_i]) \cup P^{-1}([p_i, p_{i+1}])$, the union of two vertical, infinite bands.

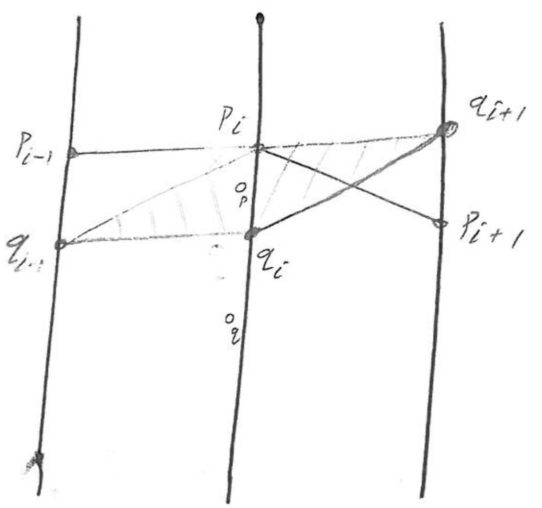


- Move p_i to q_i while fixing p_j and q_j for $i \neq j$.
- First, preform an e.d. for K along $\Delta\, p_{i-1}, q_i, p_i$ to form K'
- Second, preform an e.d. for K' along $\Delta\, p_i, q_i, p_{i+1}$

Case 2 Suppose $P([p_{i-1}, p_i] \cup [p_i, p_{i+1}])$ meets exactly one crossing. Then there is a sequence of two elementary deformations taking p_i to q_i or q_i to q_i (but not both).

Examine $P^{-1}([p_{i-1}, p_i]) \cup P^{-1}([p_i, p_{i+1}])$ and assume we see a under crossing on $P^{-1}([p_{i-1}, p_i])$. (i.e. 

)

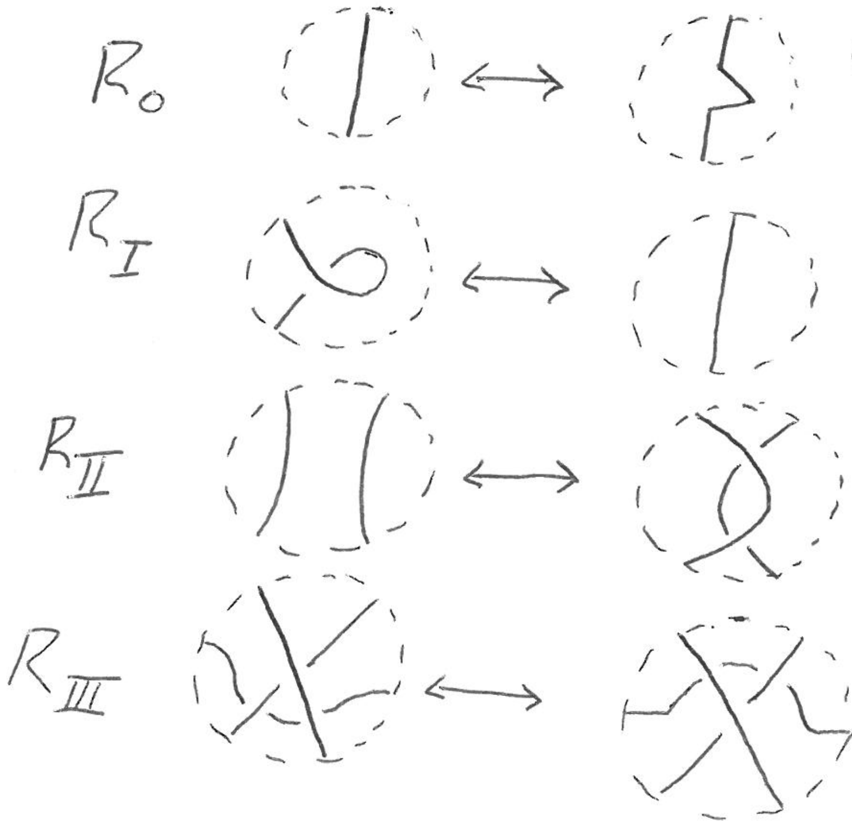


Move q_i to p_i.

1. Apply an e.d. to J along $\Delta\, q_i\, p_i\, q_{i-1}$
2. Apply an e.d. along $\Delta\, q_i, p_i, q_{i+1}$.

Hence, we found a sequence of e.d. taking K to J. $\square$

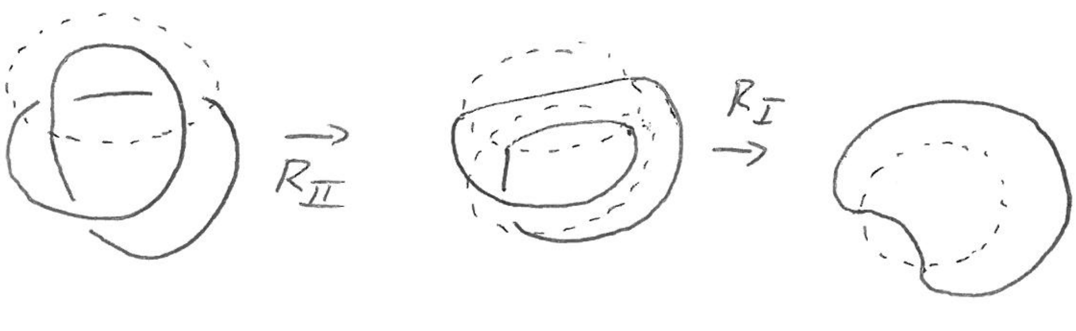
Reidermeister moves for knot diagrams



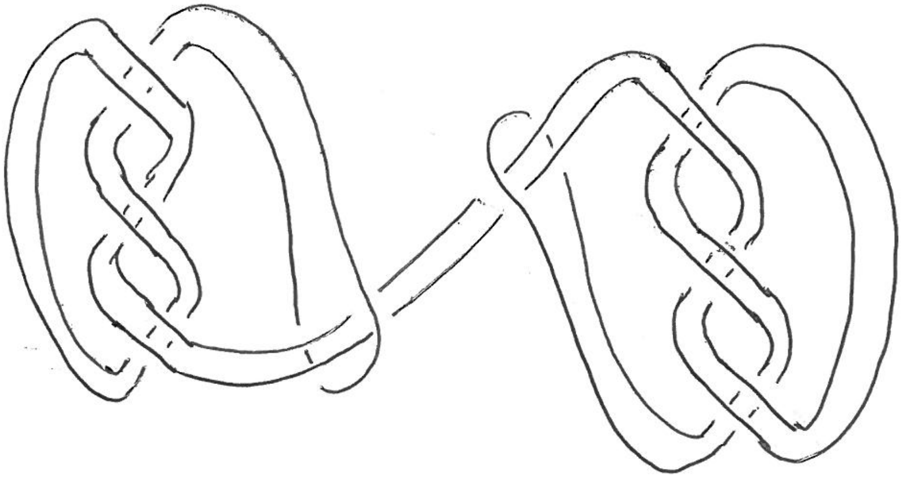
(this is the projection of an elementary deformation)

Th^m (Alexander & Briggs)
If two knots are equivalent, their diagrams are related by a sequence of Reidermeister moves.

Ex



Monster unknots
"Cousin it"



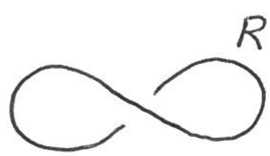
Th^m (Hass and Lagaris)
Any unknotted knot diagram D with n crossings can be transformed to the trivial knot diagram using at most $2^{(10^{11})n}$ Reidermeister moves.

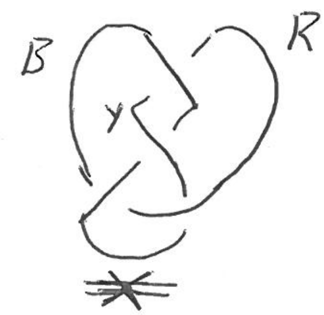
Colorability of knots (Due to Ralf Fox)

Def| A knot diagram is <u>colorable</u> if each arc can be drawn using one of three collors (red, yellow, blue) s.t.

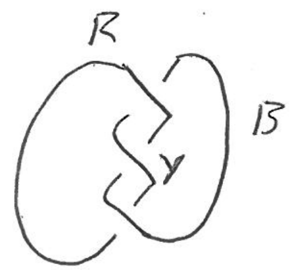
1) At least two colors are used
2) at any crossing where two colors ~~are~~ appear, all three appear

Ex| Not colorable





Colorable

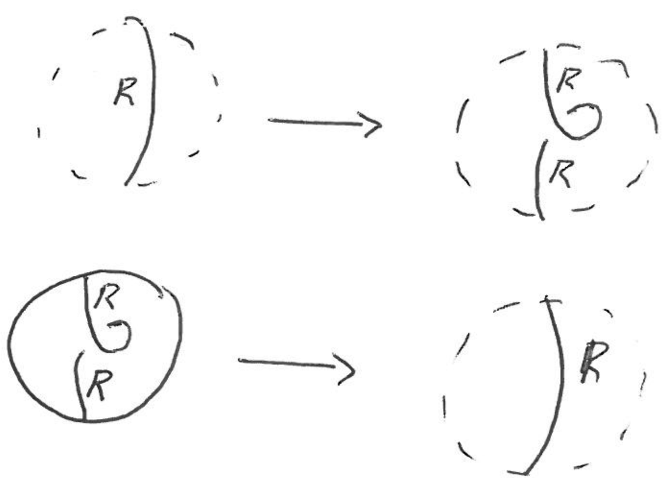


By the theorem of Alexander and Briggs, Collorability is a <u>knot</u> <u>invariant</u> if it is preserved under each of the Reidermeister moves.

Claim 1: Colorability is preserved under R_0

Clearly true

Claim 2: Colorability is preserved under R_I



Since at least 2 colors are used and since the R_I move preserves the colors disjiont from a nbh of the R_I move, then the R_I move preserves colorability.