Knot Theory Day 19

- Outline
- Review of Last time
- $g(K_1 \# K_2) = g(K_1) + g(K_2)$

- Announcements
  - Exam on Thursday
  - Extra O.H. Thursday Morning 10am to noon.
  - 5 questions
    1 question of definitions.

Recall:

F_1 $\quad$ F_2 $\quad$ $F_1 \#_\partial F_2$

If $\partial F_1 = K_1$ and $\partial F_2 = K_2$ then $\partial(F_1 \#_\partial F_2) = K_1 \# K_2$

Goal: Show $g(K_1) + g(K_2) = g(K_1 \# K_2)$

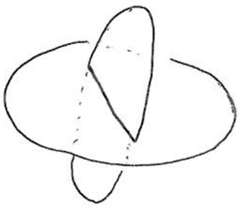
Tools:

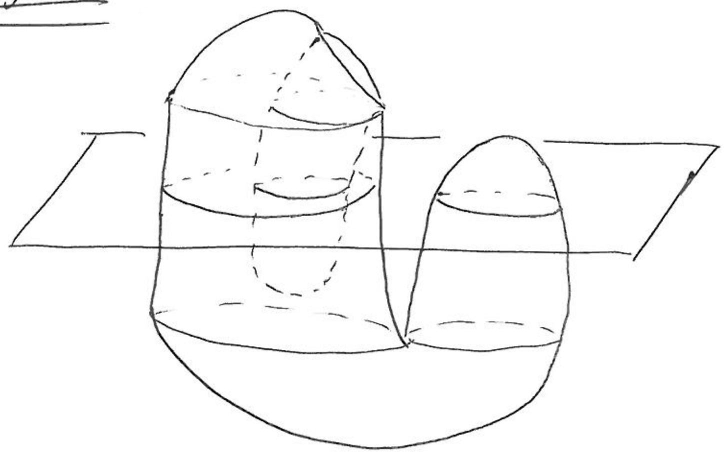
Thm

① If F' is connected $g(F') = g(F) - 1$

② If $F' = F'' \amalg F'''$, then $g(F) = g(F'') + g(F''')$

Thm | Given surfaces F_1 and F_2 embedded in $\mathbb{R}^3$, there is a small isotopy of F_i after which F_1 intersects F_2 "transversely" in disjiont loops and arcs.





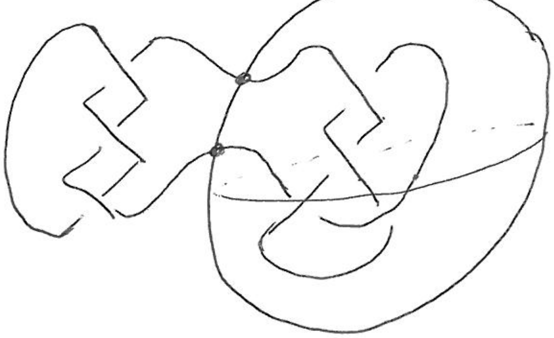
Thm | $g(K_1 \# K_2) = g(K_1) + g(K_2)$

Pf | Last time

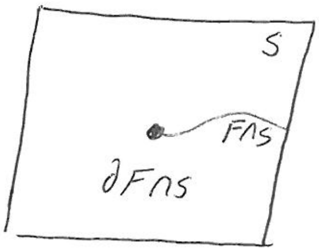
$$g(K_1 \# K_2) \leq g(K_1) + g(K_2)$$

WTS $g(K_1 \# K_2) \geq g(K_1) + g(K_2)$

Suppose F is a genus minimizing seifert surface for $K_1 \# K_2$. Let S be the "separating sphere" for $K_1 \# K_2$.

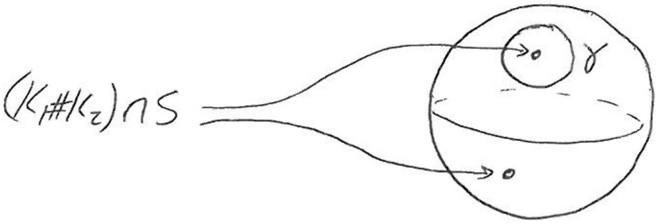


- Hence $F \cap S$ is a collection of arcs & loops.
- Since S has no boundary and ∂F intersects S in exactly two points, then $F \cap S$ contains exactly one arc.



- Claim: No loop of $F \cap S$ separates a point in $S \cap (K_1 \cap K_2)$ from another point in $S \cap (K_1 \# K_2)$

<u>Pf</u> Suppose, to form a contradiction, that such a loop γ exists. Let α be the arc of $F \cap S$ with endpoints $(K_1 \# K_2) \cap S$. Since γ separates these points $\alpha \cap \gamma \neq \emptyset$. This contradicts $F \cap S$ being a collection of <u>disjiont</u> arcs and loops. $\square$



Hence, every loop of $F \cap S$ bounds a disk in S disjiont from $K_1 \# K_2$. Let γ be an "innermost" such loop.
(i.e.) γ bounds a disk D in S s.t. $int(D) \cap F = \emptyset$.
So, D is a <u>surgery disk</u> for F.

Let F' be the result of surgery on F along D.

If F' has two components, relable F' as the component with boundary $K_1 \# K_2$. By our theorem $g(F') \leq g(F) = g(K_1 \# K_2)$. Hence, F' is a minimal genus seifert surface intersecting S in one fewer loop. Repeat this process to produce a minimal genus seifert surface F^* that meets S in a single arc. Hence, $F^* = F_1 \#_\partial F_2$ for F_i a seifert surface for K_i. Thus

$$g(K_1 \# K_2) \geq g(F_1) + g(F_2) \geq g(K_1) + g(K_2).$$