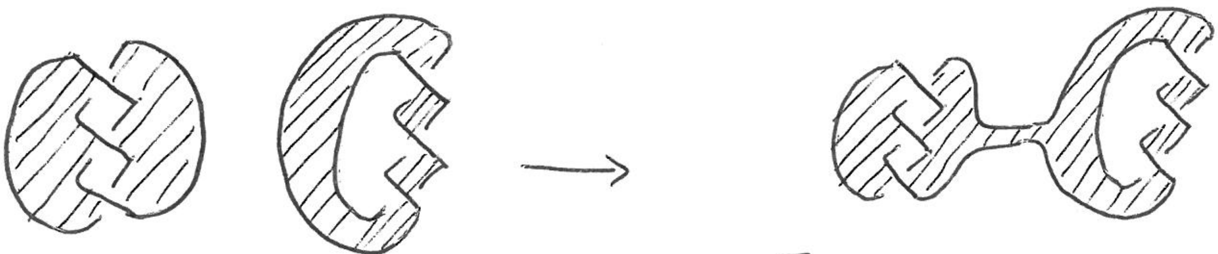
Knot theory Day 18

Outline

- genus is additive

Recall from last time



F_1 $\quad$ F_2 $\quad\quad$ $F_1 \#_\partial F_2$

<u>Lemma</u> $g(F_1 \#_\partial F_2) = g(F_1) + g(F_2)$

<u>Thm</u> Suppose F is a connected orientable surface with $C \subset F$ a knot s.t. C bounds ~~a dis~~ an embedded disk D s.t. $int(D) \cap F = \emptyset$. Let F' be the surface that results from surjery of F along D.

1. If F' is connected $g(F') = g(F) - 1$
2. If $F' = F'' \amalg F'''$ then $g(F) = g(F'') + g(F''')$.

Thm $g(K_1 \# K_2) = g(K_1) + g(K_2)$.

Pf First show $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$

Let F_1 be a minimal genus Seifert surface for K_1 and let F_2 be a minimal genus Seifert surface for K_2.

Since $\partial(F_1 \#_\partial F_2) \cong K_1 \# K_2$, then

$F_1 \#_\partial F_2$ is a Seifert surface for $K_1 \# K_2$.

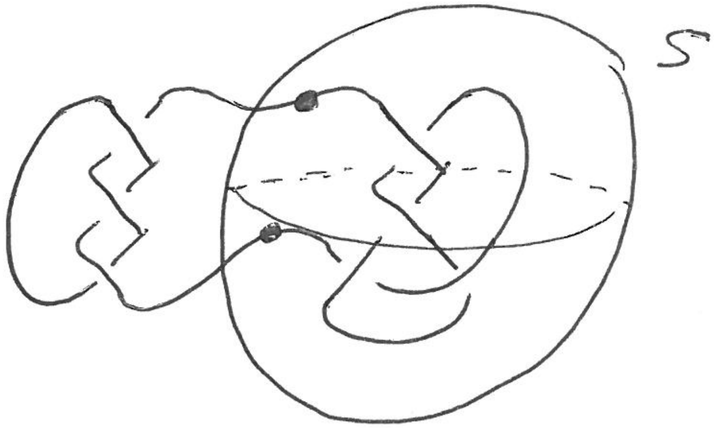
By our lemma $g(F_1 \#_\partial F_2) = g(F_1) + g(F_2)$.

Hence, ~~the~~

$$\begin{aligned} g(K_1 \# K_2) &\leq g(F_1 \#_\partial F_2) \\ &\leq g(F_1) + g(F_2) \\ &\leq g(K_1) + g(K_2). \end{aligned}$$

Second, show $g(K_1 \# K_2) \geq g(K_1) + g(K_2)$.

Let F be a minimal genus Seifert surface for $K_1 \# K_2$. Let S be a separating sphere for the connected sum.



Examine $F \cap S$.

<u>Th^m</u>| Given surfaces F_1 and F_2 embedded in $\mathbb{R}^3$, there exists an arbitrarily small isotopy of F_i after which $F_1 \cap F_2$ is a collection of arcs and loops embedded in both F_1 and F_2.

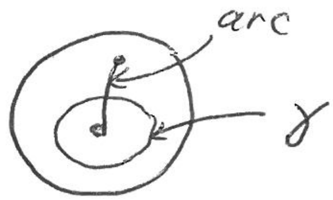
Hence, $F \cap S$ is a collection of arcs and loops. Since S is boundary less and ∂F meets S in exactly two points, then $F \cap S$ consists of one arc and a collection of loops.

Suppose $S \cap F$ contains a loop of intersection

Claim: No loop of $S \cap F$ separates a point in $S \cap (K_1 \# K_2)$ from ~~anot~~ the other point in $S \cap (K_1 \# K_2)$ is S.

<u>Pf</u>| $\exists$ $\gamma \subset S \cap F$ does separate these two points.

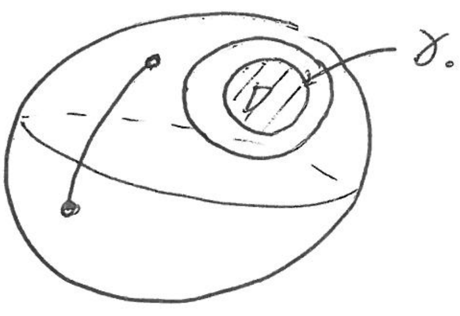
Then γ must intersect the arc of intersection in $F \cap S$.



This contradicts our theorem regarding $F \cap S$. $\square$

Hence every loop ~~in~~ of $F \cap S$ in S bounds a disk disjiont from $K_1 \# K_2$ on one side and a disk meeting $K_1 \# K_2$ in two points on the other side.

Let γ be an innermost loop of $F \cap S$ in S bounding a disk D in S disjiont from $K_1 \# K_2$



Since $\text{int}(D) \cap F = \emptyset$

After surgering F allong D to produce a surface F_1, we know F' is a Seifert surface for $K_1 \# K_2$. By our Thm from last time $g(F')$ (or the component of F' that contains $K_1 \# K_2$) is less then or equal to $g(F)$.

Additionally, $F' \cap S$ consists of at least one fewer arc (having eliminated γ).

Repeat this process to produce a Seifert surface F^* for $K_1 \# K_2$ s.t. $g(F^*) \leq g(F)$ and $F^* \cap S$ consists of a single arc.

However, since $F^* \cap S$ is a single arc $F^* = F_1 \#_\partial F_2$ for some ~~seig~~ Seifert surfaces F_1 of K_1 and F_2 of K_2.

Thus $g(F) \geq g(F^*) = g(F_1) + g(F_2) \geq g(K_1) + g(K_2) \geq g(K_1 \# K_2)$ $\square$.