Knot Theory Day 16

Out line

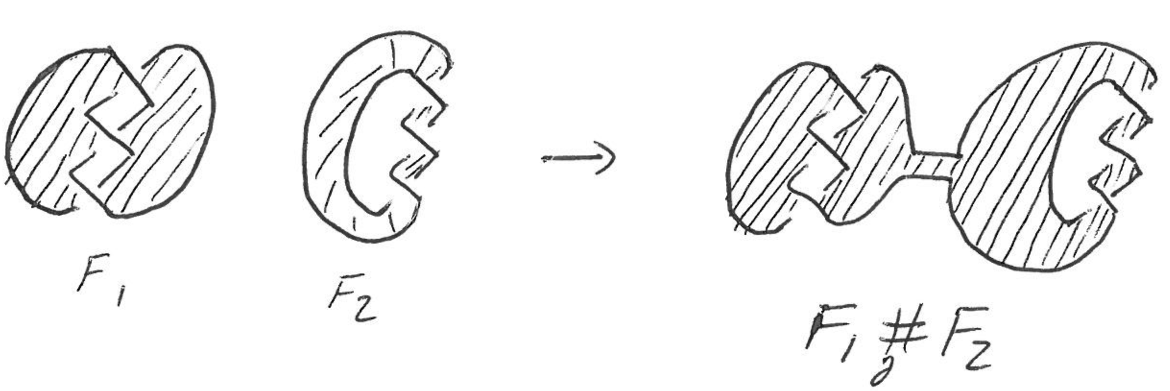
- genus of a knot
- surgery on surfaces

Recall A Seifert surface for a knot K is an embedded, orientable, polyhedral surface F s.t. $\partial F = K$.

Last time we saw every knot has a Seifert surface. (Seifert's Algorithm)

Def The genus of a knot is the minimal genus of any Seifert surface for that knot.

Def Given two surfaces with boundary F_1 and F_2 the boundary connected sum $F_1 \#_\partial F_2$ is the surface with boundary formed by identifying an arc in ∂F_1 with an arc in ∂F_2



Lemma $g(F_1 \#_\partial F_2) = g(F_1) + g(F_2)$.

Pf Recall $g(F) = \dfrac{2 - \chi(F) - B}{2}$.

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

$$\chi(F_1 \#_\partial F_2) = \chi(F_1) + \chi(F_2) - \chi(\text{—})$$

$$= \chi(F_1) + \chi(F_2) - 1$$

$$B_\# = B_{F_1} + B_{F_2} - 1$$

$$g(F_1 \#_\partial F_2) = \frac{2 - (\chi(F_1 \#_\partial F_2)) - B_\#}{2}$$

$$= \frac{2 - \chi(F_1) - \chi(F_2) + 1 - B_{F_1} - B_{F_2} + 1}{2}$$

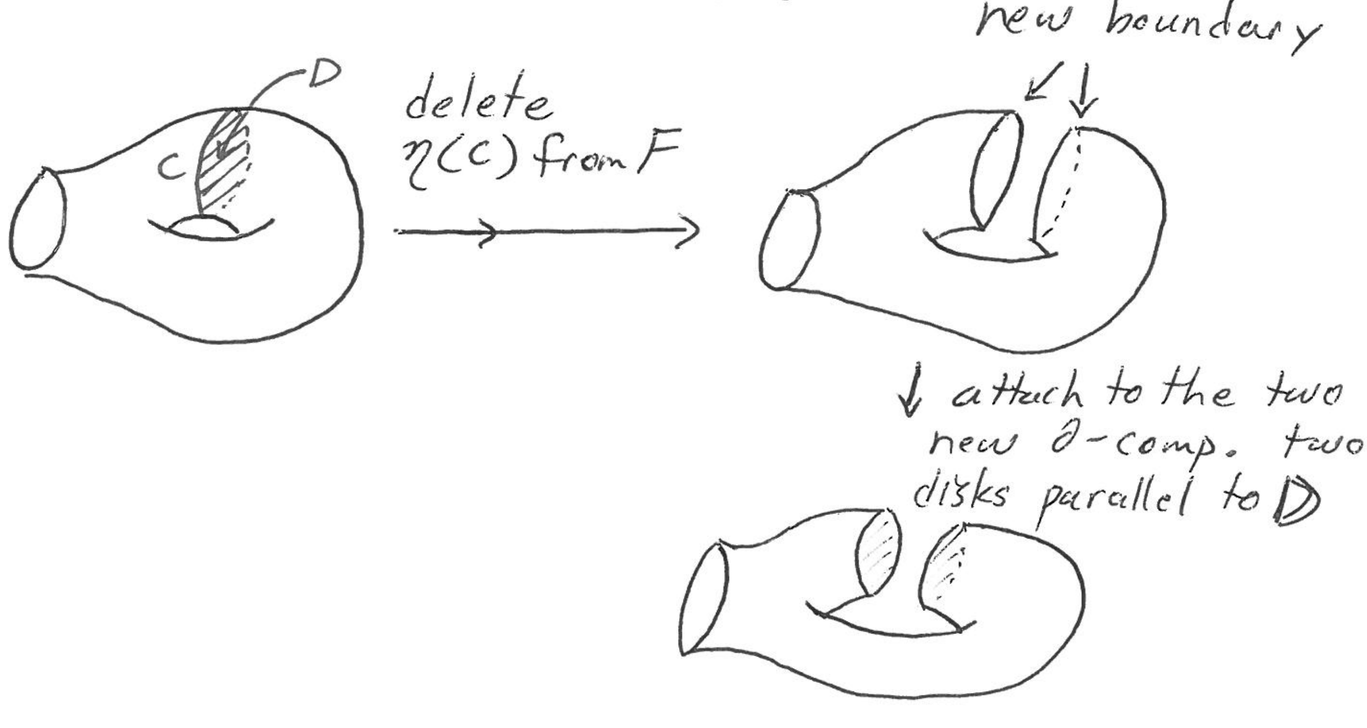
$$= \frac{2 - \chi(F_1) - B_{F_1}}{2} + \frac{2 - \chi(F_2) - B_{F_2}}{2}$$

$$= g(F_1) + g(F_2). \quad \square$$

Def Given a poly. surface F embedded in $\mathbb{R}^3$ if there exists a ~~curve~~ knot $C \subset F$ s.t. C bounds an embedded disk D s.t. $\operatorname{int}(D) \cap F = \emptyset$, then F is <u>compressible</u> and D is a <u>compressing disk for F</u>.

If F is not compressible we say F is <u>incompressible</u>. (perhaps the single most important concept in the study of 3-manifolds).

If F is compressible with compressing disk D we can <u>surger F along D</u> to form a new embedded surface F'



Thm | Suppose F is a connected orientable surface that is compressible with compressing disk D. Let F' be the surface that results from surgery of F along D.

① If F' is connected, then $g(F') = g(F) - 1$

② If $F' = F'' \amalg F'''$, then $g(F) = g(F'') + g(F''')$.

Pf |
$$\chi(F) = \chi(F - \text{annulus}) + \chi(\text{annulus}) - \chi(S^1 \amalg S^1)$$
$$= \chi(F - \text{annulus})$$
$$\chi(F') = \chi(F - \text{annulus}) + \chi(D^2 \amalg D^2) - \chi(S^1 \amalg S^1)$$
$$\boxed{\chi(F') = \chi(F) + 2}$$

$$g(F_1) = \frac{2 - \chi(F_1) - B}{2}$$
$$= \frac{2 - (\chi(F) + 2) - B}{2}$$
$$= \frac{2 - \chi(F) - B}{2} - 1$$
$$\boxed{g(F_1) = g(F) - 1}$$

$\boxed{\text{Alternatively}}$ if $F' = F'' \amalg F'''$

~~genus~~ $g(F'') + g(F''') = \dfrac{2-\chi(F'')-B_1}{2} + \dfrac{2-\chi(F''')-B_2}{2}$

$$= \frac{4-(\chi(F'')+\chi(F'''))-(B_1+B_2)}{2}$$

$$= \frac{4-(\chi(F'))-B}{2}$$

$$= \frac{2-\chi(F)-B}{2}$$

$$= g(F)$$