Knot Theory Day 16

Outline
- genus of a knot
- surgery on surfaces

Recall: A Seifert surface for a knot $K$ is an embedded, orientable, polyhedral surface $F$ s.t. $\partial F = K$.

Last time we saw every knot has a Seifert surface. (Seifert's Algorithm)

**Def.** The genus of a knot is the minimal genus of any Seifert surface for that knot.

**Def.** Given two surfaces with boundary $F_1$ and $F_2$ the boundary connected sum $F_1 \# F_2$ is the surface with boundary formed by identifying an arc in $\partial F_1$ with an arc in $\partial F_2$.
Lemma \( g(F, \# F) = g(F_1) + g(F_2). \)

**Proof**

Recall \( g(F) = \frac{2-X(F)-B}{2}. \)

\[
X(A \cup B) = X(A) + X(B) - X(A \cap B) \]

\[
X(F_1 \# F_2) = X(F_1) + X(F_2) - X(-) \]

\[
= X(F_1) + X(F_2) - 1 \]

\[
B_\\# = B_{F_1} + B_{F_2} - 1 \]

\[
g(F_1 \# F_2) = \frac{2 - X(F_1 \# F_2) - B_\\#}{2} \]

\[
= \frac{2}{2} - \frac{X(F_1) + X(F_2) - 1 - B_{F_1} - B_{F_2} + 1}{2} \]

\[
= \frac{2 - X(F_1) - B_{F_1}}{2} + \frac{2 - X(F_2) - B_{F_2}}{2} \]

\[
= g(F_1) + g(F_2). \Box \]
Def: Given a poly. surface $F$ embedded in $\mathbb{R}^3$ if there exists a curve knot $C \subset F$ s.t. $C$ bounds an embedded disk $D$ s.t. $\text{int}(D) \cap F = \emptyset$, then $F$ is compressible and $D$ is a compressing disk for $F$.

If $F$ is not compressible we say $F$ is incompressible. (perhaps the single most important concept in the study of 3-manifolds).

If $F$ is compressible with compressing disk $D$ we can surger $F$ along $D$ to form a new embedded surface $F'$.

diagram:
- delete $\eta(C)$ from $F$
- attach to the two new $\partial$-components two disks parallel to $D$
Thm
Suppose $F$ is a connected orientable surface that is compressible with compressing disk $D$. Let $F'$ be the surface that results from surgery of $F$ along $D$.

1. If $F'$ is connected, then $g(F') = g(F) - 1$

2. If $F' \neq F'' \parallel F'''$, then $g(F) = g(F'') + g(F''')$.

PF

$\chi(F) = \chi(F - \text{annulus}) + \chi(\text{annulus}) - \chi(S^1 \times S^1)$

$= \chi(F - \text{annulus})$

$\chi(F') = \chi(F - \text{annulus}) + \chi(D^2 \cup D^2) - \chi(S^1 \times S^1)$

$\boxed{\chi(F') = \chi(F) + 2}$

$g(F_i) = \frac{2 - \chi(F_i)}{2} - B$

$= \frac{2 - (\chi(F) + 2)}{2} - B$

$= \frac{2 - \chi(F)}{2} - 1$

$g(F_i) = g(F) - 1$
Alternatively, if $F' = F'' \cup F'''$

Then genus $g(F'') + g(F''') = \frac{2 - \chi(F'') - B_i}{2} + \frac{2 - \chi(F''') - B_i}{2}$

\[= \frac{4 - (\chi(F'') + \chi(F''')) - (B_i + B_i)}{2}\]

\[= 4 - \left(\chi(F')\right)^2 - B\]

\[= \frac{Z - \chi(F)}{2}\]

\[= g(F)\]