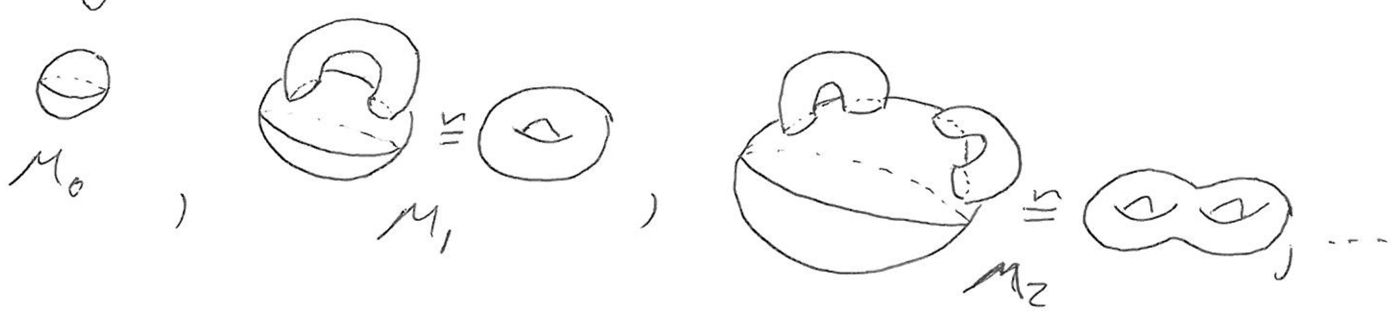
Knot Theory Day 16

Out line
- Seifert surfaces
- genus of a knot

Wrapping up last time

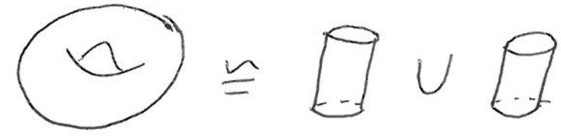
Final classification theorem

Thm Any closed connected orientable surface is homeomorphic to exactly one M_g ($g = 0, 1, \ldots$ "a sphere with g handles"). (here g indicates genus).



Def The genus of a connected orientable surface S is given by
$$g(S) = \frac{2 - \chi(S) - B}{2}$$
where B is the the number of components in ∂S.

Ex Cacalulate $g($$)$



From last time

$\chi($$) = \chi($$) + \chi($$)$
$- \chi(\circ\circ)$
$= 0 + 0 - 0 = 0$

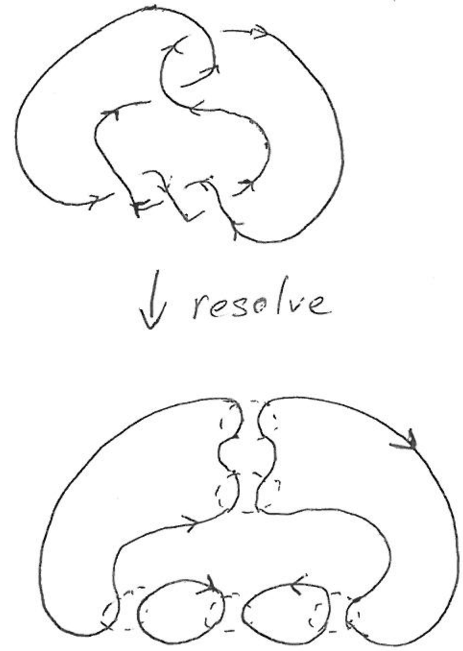
So $g(\;) = \dfrac{2-0-0}{2} = 1.$

<u>Def</u> A Seifert Surface for a knot K in $\mathbb{R}^3$ is a connected orientable polyhedral surface embedded in $\mathbb{R}^3$ with boundary K.

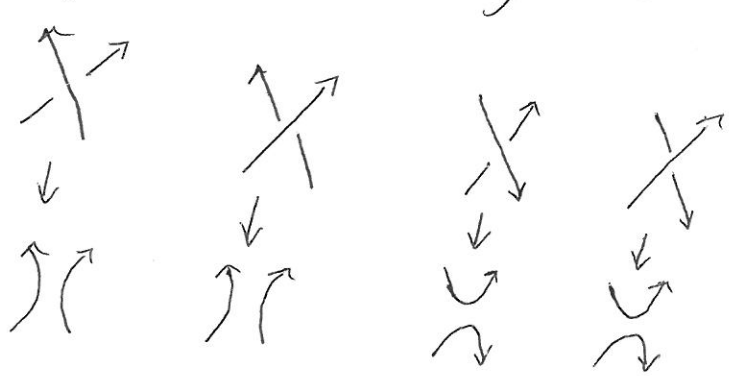
<u>Thm</u> (Seifert) Every knot has a Seifert surface.

<u>Pf</u> We will estabish an algorithm for generating such a surface from any knot diagram.

Let D be an oriented knot diagram of a knot K.

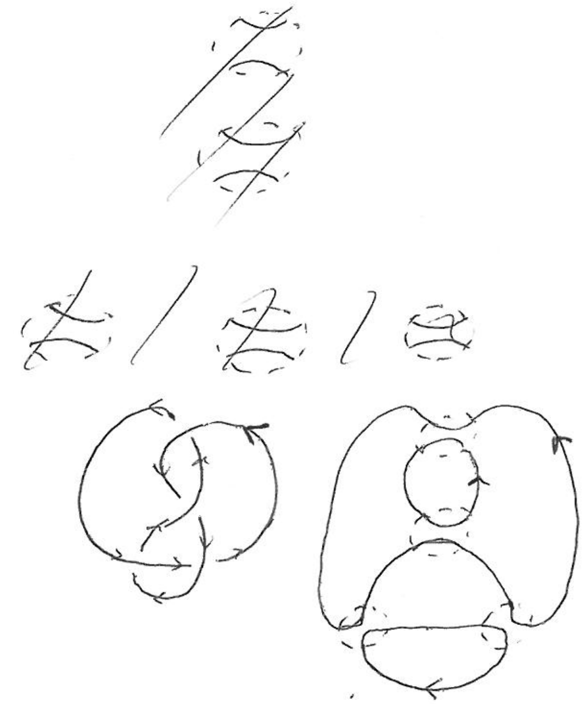


Step 1: We resolve each crossing of D using the following rules

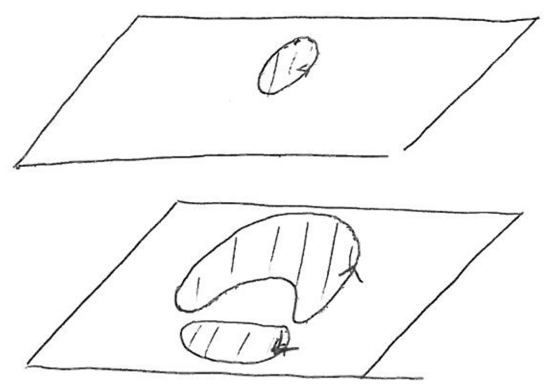


(i.e. so orientations stay consistent).

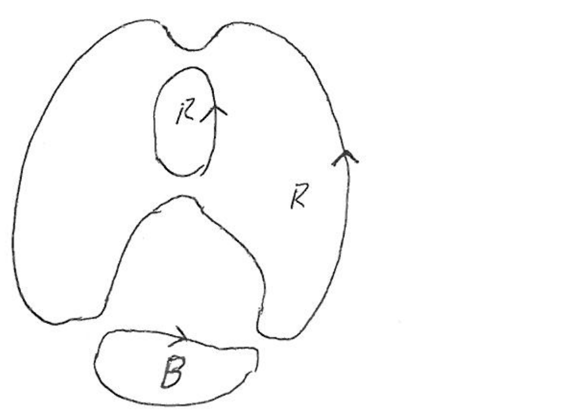
Step 2 | Let each loop in the resolution bound a disk in a plane parallel to the plane of projection so that all disks are disjiont.



If there are nested circles we embedded the disks ~~of~~ ~~each~~ bounding inner circles on sequentially higher planes.



Step 3 | Orient each disk s.t. ~~the~~ each has one side colored red and one side colored blue. The red side faces up if the boundary is oriented counter clockwise and the red side faces down if the boundary is oriented clockwise



Step 4 At each crossing attach a half twisted band (with twisting ~~consiste~~ chosen to be consistant with the crossing) to ~~ea th~~ each of the boundaries of the disks incident to the crossing. Notice that in each case there is a choice of orientation on the band that is consistent with the existing orientations on the disks.

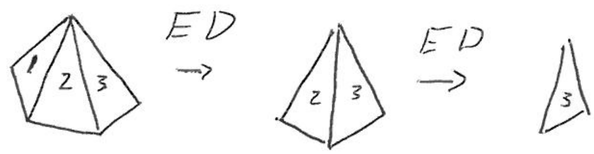
Hence, we have produced an embedded polyhedral orientable surface with K as its boundary.

<u>Def</u> The <u>genus of a knot</u> K is the minimal genus of any Seifert surface for K.

<u>Thm</u> If K bounds a disk then K is the unknot.

<u>Pf</u> Suppose K is a knot bounding an embedded polyhedral surface F s.t. F is a disk. After ordering the triangles of F appropriately, these triangles define a sequence of elementary deformations. Hence K is equivalent to a knot that is the boundary of a metric triangle.

i.e.



<u>Exercise</u> If F is a Seifert surface for a knot

$$g(F) = \frac{2-\chi(F)-1}{2} = \frac{1}{2} - \frac{1}{2}\chi(F)$$

$$\chi(F) = \chi(\text{Seifert disks}) + \chi(\text{bands}) - \chi(\partial \text{ bands})$$

$$= \#\text{ of seifert circles} + \#\text{of crossings} - 2\#\text{of crossings}$$

$$= \#\text{of seifert circles} - \#\text{of crossings}.$$

Ex | Find the genus of the trefoil

$$\chi(F) = 2 - 3 = -1$$

$$g(F) = \frac{1}{2} - \frac{1}{2}\chi(F) = \frac{1}{2} - \frac{1}{2}(-1) = 1$$

~~Since~~ Hence, $g(\text{trefoil}) = 1$.