Knot theory Day 14

Outline

- Combinatorial Surfaces
- Euler charicteristic

Last time we defined n-manifold

Topological Def| A top. space X is an n-manifold if X is second Countable, Hausdorff and locally homeomorphic to $\mathbb{H}^n$.

Given three points in $\underbrace{\mathbb{R}^3}_{}$ we can define the metric triangle defined by those points as

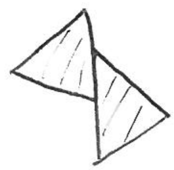
$$\Delta P_1P_2P_3 = \{ xP_1 + yP_2 + zP_3 \mid x+y+z=1 \text{ and } x,y,z \geq 0 \}$$

Def| A polyhedral surface is the union of a finite collection of metric triangles s.t.

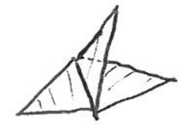
1) Each pair of triangles is disjiont or intersect in an edge or vertex.
2) At most two triangles share a common edge
3) The union of the edges that are contained in exactly one triangle is the disjiont union of simple closed polygonal curves.

What is not allowed

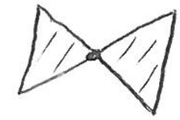
1)



2)

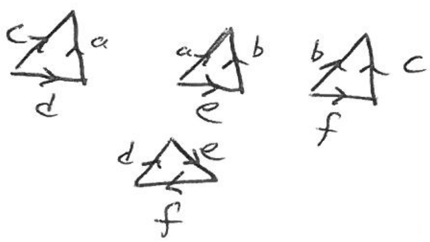


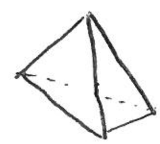
3)



Note: This is an abstract definition (i.e. unrelated to an embedding in $\mathbb{R}^n$.)

Example





This is a combinatorial surface

It can be realized as an embedded surface in $\mathbb{R}^3$. (In particular, it is a combinatorial 2-sphere)

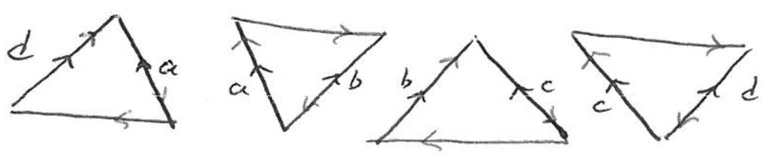
A deep result from differential topology

Thm: Every compact 2-manifold is homeomorphic to a combinatorial surface.

Orientation: In contrast to vector calc., there is an intuitive and combinatorial definition for orientation on ~~combinatorial~~ polyhedral surfaces.

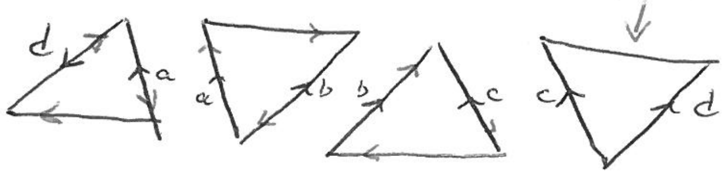
Def A polyhedral surface F is orientable if it is possible to assign an orientation to each of ~~its con triangles so~~ the boundaries of its triangles so that whenever two triangles meet in an edge, ~~the orientation~~ each of the two induced orientations on that edge are opposite.

ie.



annulus

Problem!

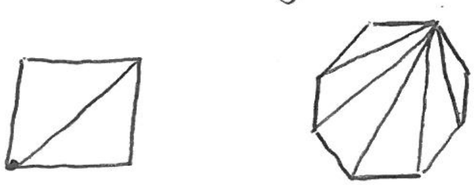


Möbius ~~Man~~ Band.

When are two combinatorial surfaces equivalent?

Note: Unlike knots this is an <u>intrinsic</u> equivalence irrespective of how they sit in space.

Ex The following should be the same



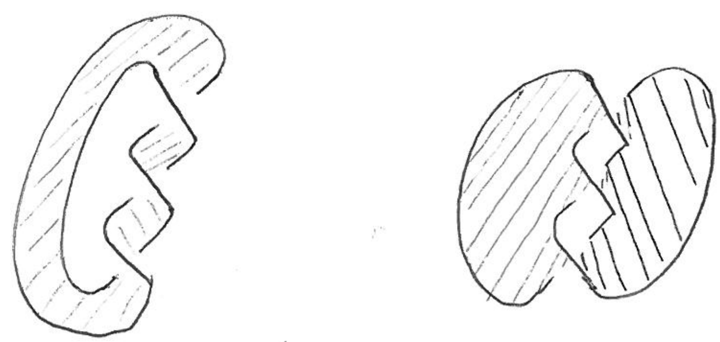
So, insisting that triangles get mapped to triangles is too restrictive.

Note We often draw embedded polyhedral surfaces as smooth (i.e. use lots of triangles).

We are building up to the following theorem

Thm Every knot bounds an orientable embedded polyhedral surface.

Example



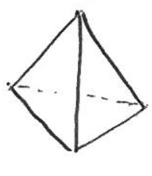
The trefoil bounds both an orientable and non orentable embeded poly hedral surface.

Fact Any embedded polyhedral surface in $\mathbb{R}^3$ bounds a knot.

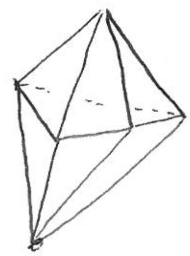
Def Given a Polyhedral surface F with f triangles e edges and v vertices (all counted after taking identifications) the Eueler characteristic is given by $f-e+v=\chi(F)$.

Deep fact from algebraic topology: The Euler charicteristic is an invariant of surfaces up to homeomorphism. (In fact, it is an invariant of symplicial complexes up to homotopy equivalence)

$\boxed{\text{Ex}}$



S^2



S^2

$v - e + f =$

$4 - 6 + 4 = \cancel{8}2$

$v - e + f$

$6 - 12 + 8 = 2$