Knottheory Day 12

Outline

- More on Kauffman bracket & crossing #

Recall:

Thm| The breadth of a Kauffman bracket ~~is~~ of a reduced alternating diagram D is $4c$ where c is the number of crossings in D.

Pf| Last time

Today|

Thm| The span of the Kauffman bracket of any diagram with c crossings is less than or equal to $4c$.

Lemma| Let D be any diagram with c crossings. Let $S_+ = S_0, S_1, S_2, \ldots, S_i$ be a sequence of states on D s.t. S_j has exactly j negative lables and S_j differs from S_{j+1} by exactly one lable, then the highest power in $\langle D|_{S_{j+1}} \rangle$ is less than or equal to the highest power in $\langle D|_{S_j} \rangle$.

Proof| Last time

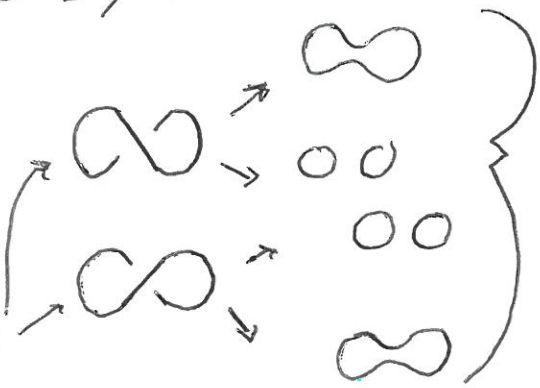
Dual state Lemma

For any state s let $\hat{s}$ denote the dual state obtained by exchanging all positive and negative labels. For any connected diagram D and any state s $|sD| + |\hat{s}D| \leq C+2$.

Pf Procede by induction on the number of crossings C

$C = 1$.

The only connected diagram is one of these



In each case $|sD| + |\hat{s}D| = 3 < 4(1)$

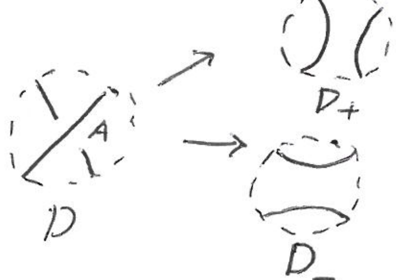
Assume the lemma is true for diagrams with $C-1$ or fewer crossings.

Let D be a connected diagram of a knot with c crossings.

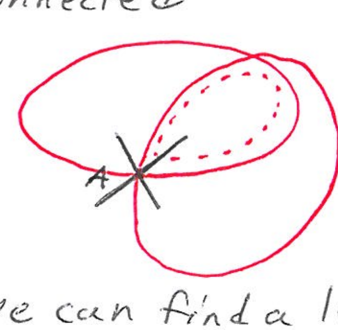
Let s be a state for D.

Let A be a crossing of D.

Let D_+, D_- be diagrams that resutt from resulving A.



If D_+ and D_- are disconnected



Then we can find a loop in the diagram that meets the diagram in exactly one point that is not a crossing *

WLOG suppose D_+ is connected and s assigns a $+1$ to A.

Let t be the restriction of s to D_+ s.t. $tD_+ = sD$.

Since $\hat{t}E$ differs from $\hat{s}D$ only at A,

$$|\hat{t}D_+| = |\hat{s}D| \pm 1.$$

By induction $|tD_+| + |\hat{t}D_+| \leq c+1$.

So $$|sD| + (|\hat{s}D| \pm 1) \leq c+1$$

Hence $$|sD| + |\hat{s}D| \leq c+2. \quad \square$$

<u>Thm</u> The span of the Kauffman bracket of any diagram D with c crossings is less than or equal to $4c$.

<u>Pf</u> By lemma 2, the largest power that might occur in $\langle D \rangle$ is the highest power in $\langle D|s_+\rangle$

$$\langle D|s_+\rangle = A^{\Sigma s_+}(-A^2 - A^{-2})^{|s_+D|-1}.$$

~~Hs~~ Highest power is $c + 2|s_+D| - 2$

Similarly, the lowest power that might occur is the lowest power in $\langle D|s_-\rangle$, $-c - 2|s_-D| + 2$

Hence, the breadth is <u>at most</u>

By Dual state lemma.

$$2c + 2(|s_+D| + |s_-D|) - 4 \leq 2c + 2(c+2) - 4 = 4c \quad \square$$

Thm 1 | If D is a reduced alternating diagram with c crossings, then the breadth of $\langle D \rangle$ is $4c$.

Thm 2 | If D is any knot diagram with c crossings then the breadth of $\langle D \rangle$ is at most $4c$.

Cor | If D is a reduced alternating diagram of a knot K with c crossings, then $c(K) = c$.

Pf | By thm 1 the breadth of $\langle D \rangle$ is $4c$. Since $f_D(A) = A^{-w(D)} \langle D \rangle$ is a knot invariant, then the breadth of $\langle D \rangle$ is a knot invariant. Suppose D' is a diagram of K with fewer than c crossings. By Thm 2, the breadth of $\langle D' \rangle$ is less than $4c$, a contradiction. Hence, every diagram of K has at least c crossings. $\square$

Thm | If K_1 and K_2 are alternating knots, the $c(K_1 \# K_2) = c(K_1) + c(K_2)$

Knot theory Day 13

Outline
- Manifolds
- Surfaces
- Combinatorial Surfaces
- Euler Characteristic.

Def A metric on a set X is a function $d: X \times X \to \mathbb{R}$
s.t.
1. $d(x,y) \geq 0$ for all $x, y \in X$
2. $d(x,y) = 0$ iff $x = y$
3. $d(x,y) = d(y,x)$ for all $x, y \in X$
4. $d(x,z) \leq d(x,y) + d(y,z)$

The pair (X, d) is a metric space.

Def $F: (X, d_x) \to (Y, d_y)$ is continuous if
$\forall x_0 \in X$ and $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.
$$d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), F(x_0)) < \varepsilon$$

Note if $A \subset X$ and (X, d) is a metric space then (A, d) is a metric subspace of X.

Def ~~ ~~ $F: (X, d_X) \to (Y, d_Y)$ is a homeomorphism if F is continuos, bijective and F^{-1} is continuous.

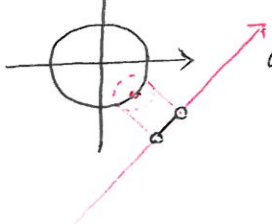
We are interested in looking at manifolds up to homeomorphism.

<u>Def</u> ~~A manifold is a~~ A <u>closed</u> n-dim manifold is a "second countable" metric space that is locally homeomorphic to $\mathbb{R}^n$.

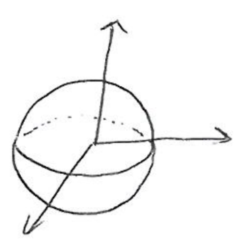
Locally Homeomorphic to $\mathbb{R}^n$

$\forall x \in X \quad \exists \varepsilon > 0$ s.t. $B_\varepsilon(x)$ is homeomorphic to an open ball in $\mathbb{R}^n$.

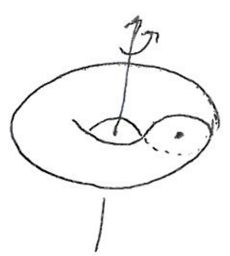
Examples of closed manifolds.



unit circle in the plane is a closed 1-manifold



unit sphere in $\mathbb{R}^{n+1}$ is an closed n-manifold



torus is a closed 2-manifold.

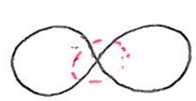


Figure 8 is <u>not</u> a closed manifold

Def | A ~~manifol~~ n-dim. manifold is a second contable metric space that is locally homeomorphic to $\mathbb{R}^n$ or upper half space.

Locally homeomorphic to $\mathbb{R}^n$ or upper half space

$\forall x \in X \quad \exists \; \varepsilon > 0$ s.t. $B_\varepsilon(x)$ is homeomorphic to an open ball in $\mathbb{R}^n$ or

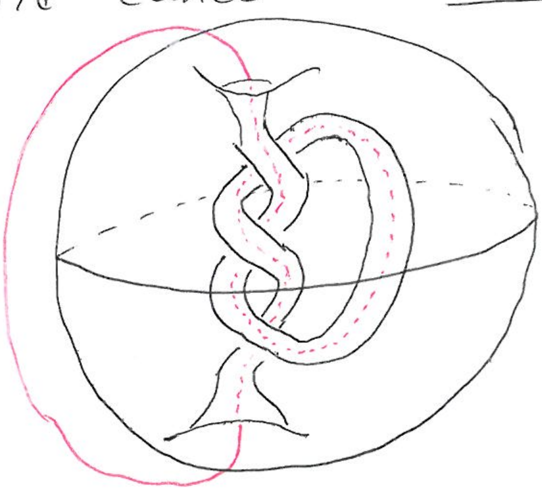
an open ball $B'_\delta(\vec{a})$ in $\{(x,y,z) \in \mathbb{R}^n \,/\, z \geq 0\} = \mathbb{H}^n$ s.t. $\vec{a} = (x_0, y_0, 0)$.

Examples

- line segments are 1-manifolds

- The unit disk in $\mathbb{R}^n$ is a n manifold.

- Let K be a knot in S^3 (the unit sphere in $\mathbb{R}^4$)
Let $\eta_\varepsilon(K)$ be the set of all points in S^3 within ε of K. For ε small $S^3 - \eta_\varepsilon(K)$ is a 3-manifold called the <u>knot</u> <u>exterior</u>.



← The exterior of the trefoil embedded in $\mathbb{R}^3$.

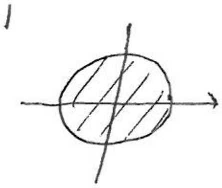
Def

Given an n-manifold M, ∂M is the set of all points in M that fail the locally homeomorphic to $\mathbb{R}^n$ condition.

Examples

If M is a line segment, ∂M is the set of the two endpoints of M.

If M is a unit disk in $\mathbb{R}^n$, ∂M is S^{n-1}.



If M is a knot exterior ∂M is a

knot exterior, ∂M is a torus (i.e. $T^2 \cong S^1 \times S^1$).

Thm If M is an n-manifold, ∂M is a closed $(n-1)$-manifold.

(i.e. $\partial(\partial M) = \emptyset$).

Whitney embedding Thm | Every m-manifold can be embedded in $\mathbb{R}^{2m}$.

Def | ~~B~~ Y can be embedded in X if there exists a function $F: Y \to X$ s.t. F is a "nice" homeomorphism onto its image.

Examples | Knots are embeddings of S^1 into $\mathbb{R}^3$

Def | A <u>surface</u> is a 2-manifold

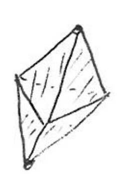
Def | A <u>polyhedral surface</u> is a a finite union of metric triangles (i.e. $\Delta P_1P_2P_3 \{ xp_1 + yp_2 + zp_3 \mid x+y+z=1,\ x,y,z \geq 0 \}$)

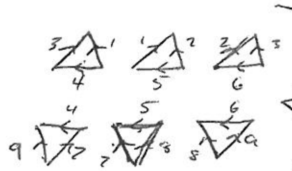
s.t.

1) each pair of triangles is either disjiont or their intersection is a common edge or vertex
2) at most two triangles share a common edge
3) The union of all edges that are contained in exactly one triangle is a disjiont collection of simple polygonal curves.

Examples

 a polygonal disk

 a polygonal sphere



6 tringles plus the nessesary glueing information to build a 2-sphere.

Thm| Every surface is homeomorphic to a polygonal surface.

Pf| Deep result of differential topology.