Outline
- More on Kauffman bracket & crossing #

Recall:
Thm. The breadth of a Kauffman bracket of a reduced alternating diagram $D$ is $4c$ where $c$ is the number of crossings in $D$.

Pf. Lost time

Today
Thm. The span of the Kauffman bracket of any diagram with $c$ crossings is less than or equal to $4c$.

Lemma 2. Let $D$ be any diagram with $c$ crossings. Let $S_1^+, S_0^+, S_1^-, S_0^-$ be a sequence of states on $D$ s.t. $S_j^-$ has exactly $j$ negative labels and $S_j^+$ differs from $S_{j+1}^+$ by exactly one label, then the highest power in $<D|S_{j+1}^+>$ is less than or equal to the highest power in $<D|S_j^->$.

Pf. Last time
Dual state Lemma. For any state \( s \) let \( \tilde{s} \) denote the dual state obtained by exchanging all positive and negative labels. For any connected diagram \( D \) and any state \( s \), \( |sD| + |\tilde{s}D| \leq C + 2 \).

Proof. Proceed by induction on the number of crossings \( C \).

- **Case 1:** \( C = 1 \).

  The only connected diagram is one of these:

  In each case, \(|sD| + |\tilde{s}D| = 3 < 4(1)\).

Assume the lemma is true for diagrams with \( C - 1 \) or fewer crossings.

Let \( D \) be a diagram of a knot with \( C \) crossings. Let \( s \) be a state for \( D \).

Let \( A \) be a crossing of \( D \).

Let \( D_+ \), \( D_- \) be diagrams that result from resolving \( A \).

If \( D_+ \) and \( D_- \) are disconnected

Then we can find a loop in the diagram that meets the diagram in exactly one point that is not a crossing.
WL0G suppose $D_+$ is connected and $s$ assigns a $+1$ to $A$.

Let $t$ be the restriction of $s$ to $D_+ \leq b. \hat{t}D_+ = sD_+$. Since $\hat{t}E$ differs from $sD$ only at $A_+$

$$\hat{E}D_+ = sD_+ + 1.$$ 

By induction $1 \leq D_+ + 1 \leq C + 1$.

So $|sD| + (|\hat{s}D| + 1) \leq C + 1$

Hence $|sD| + |\hat{s}D| \leq C + 2$. □

**Thm** The span of the Kauffman bracket of any diagram $D$ with $c$ crossings is less than or equal to $4c$.

**Pf** By lemma 2, the largest power that might occur in $<D>$ is the highest power in $<D|s_+>$

$$< D|s_+ > = A^{2s_+}(-A^2 - A^{-2})^{1|s_+D|+1}$$

Highest power is $C + 2|s_+D| - 2$

Similarly, the lowest power that might occur is the lowest power in $<D|s_->$

$$-C + 2|s_-D| + 2$$

Hence, the breadth is at most

$$2c + 2(1|s_+D| + |s_-D|) - 4 \leq 2c + 2(c + 2) - 4 = 4c$$

□
**Thm 1** If $D$ is a reduced alternating diagram with $c$ crossings, then the breadth of $\langle D \rangle$ is $4c$.

**Thm 2** If $D$ is any knot diagram with $c$ crossings then the breadth of $\langle D \rangle$ is at most $4c$.

**Cor** If $D$ is a reduced alternating diagram of a knot $K$ with $c$ crossings, then $c(K) = c$.

**Pf** By Thm 1 the breadth of $\langle D \rangle$ is $4c$.

Since $f_D(A) = A^{-w(D)} \langle D \rangle$ is a knot invariant, then the breadth of $\langle D \rangle$ is a knot invariant. Suppose $D'$ is a diagram of $K$ with fewer than $c$ crossings. By Thm 2, the breadth of $\langle D' \rangle$ is less than $4c$, a contradiction. Hence, every diagram of $K$ has at least $c$ crossings. □

**Thm** If $K_1$ and $K_2$ are alternating knots, then $c(K_1 \# K_2) = c(K_1) + c(K_2)$.
Outline
- Manifolds
- Surfaces
- Combinatorial Surfaces
- Euler Characteristic

**Def.** A metric on a set $X$ is a function $d : X \times X \rightarrow \mathbb{R}$ s.t.

1. $d(x, y) \geq 0$ for all $x, y \in X$
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) = d(y, x)$ for all $x, y \in X$
4. $d(x, z) \leq d(x, y) + d(y, z)$

The pair $(X, d)$ is a metric space.

**Def.** $F : (X, d_X) \rightarrow (Y, d_Y)$ is continuous if

$\forall x_0 \in X$ and $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$

Note if $A \subset X$ and $(X, d)$ is a metric space then $(A, d)$ is a metric subspace of $X$.

**Def.** $F : (X, d_X) \rightarrow (Y, d_Y)$ is a homeomorphism if $F$ is continuous, bijective and $F^{-1}$ is continuous.
We are interested in looking at manifolds up to 
homeomorphism.

**Def:** A manifold is a closed n-dimensional manifold
is a "second countable" metric space that is
locally homeomorphic to $\mathbb{R}^n$.

**Locally Homeomorphic to $\mathbb{R}^n$**
\[ \forall x \in X \ \exists \varepsilon > 0 \ s.t. \ B_\varepsilon(x) \text{ is homeomorphic to an open ball in } \mathbb{R}^n. \]

Examples of closed manifolds.

1. Unit circle in the plane is a 1-manifold.

2. Unit sphere in $\mathbb{R}^{n+1}$ is an n-manifold.

3. Torus is a 2-manifold.

4. Figure 8 is not a manifold.
Def: A manifold is a second countable metric space that is locally homeomorphic to $\mathbb{R}^n$ or upper half space. Locally homeomorphic to upper half space

$\forall x \in X \exists \varepsilon > 0$ s.t. $B_\varepsilon(x)$ is homeomorphic to an open ball in $\mathbb{R}^n$ or an open ball $B_\varepsilon'(\hat{a})$ in $S^3 \times \mathbb{R}^n / z = 0 \sim H^1$ s.t. $\hat{a} = (x_0, y_0, 0)$.

Examples

- Line segments are 1-manifolds
- The unit disk in $\mathbb{R}^2$ is a n manifold.
- Let K be a knot in $S^3$ (the unit sphere in $\mathbb{R}^4$)
  Let $\Omega_\varepsilon(K)$ be the set of all points in $S^3$ within $\varepsilon$ of K. For $\varepsilon < 3$, $S^3 - \Omega_\varepsilon(K)$ is a $3-$manifold called the knot exterior.
- The exterior of the trefoil embedded in $\mathbb{R}^3$. 
Def
Given an n-manifold M, ∂M is the set of all points in M that fail the locally homeomorphic to \( \mathbb{R}^n \) condition.

Examples
If M is a line segment, ∂M is the set of the two endpoints of M.

If M is a unit disk in \( \mathbb{R}^n \), ∂M is \( S^{n-1} \).

If M is a knot exterior, ∂M is a knot exterior, ∂M is a torus (i.e. \( T^2 \approx S^1 \times S^1 \)).

Thm If M is an n-manifold, ∂M is a \underline{closed} \((n-1)\)-manifold.

(i.e. \( \partial(\partial M) = \emptyset \)).
Whitney embedding Thm
Every m-manifold can be embedded in \( \mathbb{R}^{2m} \).

**Def.** \( Y \) can be embedded in \( X \) if there exists a function \( F: Y \to X \) s.t. \( F \) is a "nice" homeomorphism onto its image.

**Examples.** Knots are embeddings of \( S^1 \) into \( \mathbb{R}^3 \)

**Def.** A surface is a 2-manifold.

**Def.** A **polyhedral surface** is a a finite union of metric triangles (i.e. \( \Delta x y z : x + y + z = 1, x, y, z \geq 0 \)) s.t.

1. each pair of triangles is either disjoint or their intersection is a common edge or vertex
2. at most two triangles share a common edge
3. the union of all edges that are contained in exactly one triangle is a disjoint collection of simple polygonal curves.

**Examples**

\[ \triangle \] a polygonal disk

\[ \text{a polygonal sphere} \]

\[ \text{6 triangles plus the necessary gluing information to build a 2-sphere.} \]
Thm Every surface is homeomorphic to a polygonal surface.

Pf Deep result of differential topology.