Knot theory Day 10

Outline

- Jones Polynomial
- State sum formula for the Kauffman bracket
- Properties of the Jones Polynomial.

Recall! Given a knot diagram $D$

$$f_D(A) = (-A)^{-3w(D)} \langle D \rangle$$

is a knot invariant where $w(D)$ is the writhe and $\langle D \rangle$ is the Kauffman bracket

Def! The Jones polynomial is $f_D(A)$ where we make the substitution $A = t^{-\frac{1}{4}}$.

Von Jones won the fields medal for his work on the Jones poly. The Jones poly was a key tool use by others to solve all 3 Tait Conjectures.

State Sum model for the Kauffman bracket.
A state $s$ of a diagram $D$ is an assignment of $+1$ or $-1$ to each crossing. (A diagram has $2^{c(D)}$ states). Given a state $s$ on a diagram $D$, $sD$ denotes a resolution of $D$ into a crossingless diagram by the rule

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{image1} \\
\text{and} \\
\includegraphics[width=0.2\textwidth]{image2}
\end{array}
\end{align*}
\]

Let $|sD|$ denote the number of disjoint loops in $sD$. Let $\Sigma s$ denote the sum of the values in $s$ states.

Remarks

1) $\Sigma s$ is between $-c$ and $+c$, but has the same parity as $c$.
2) If $s$ and $t$ are states for a diagram $D$ that differ at exactly one crossing $|sD| = |tD| \pm 1$
\[
\langle D \rangle = \sum_s \langle D|s \rangle \\
\text{where the sum is over all states of } D \text{ and} \\
\langle D|s \rangle = A^S (A^2 - A^{-2})^{1_s(D)} |1 \rangle
\]

(\langle D|s \rangle \text{ is the "contribution" of the state } s \text{ to the Kauffman bracket})
Proof) - Let D be a diagram with C crossings.
- Order these crossings from 1 to C.
- Let $D_+$ denote the diagram D with crossing 1 resolved positively.
- Let $D_-$ denote diagram D with crossing 1 resolved negatively and crossing 2 resolved positively etc.

$$\langle D \rangle = A \langle D_+ \rangle + A^{-1} \langle D_- \rangle$$

$$= A (A \langle D_{++} \rangle + A^{-1} \langle D_{+-} \rangle) + A^{-1} (A \langle D_{-+} \rangle + A^{-1} \langle D_{--} \rangle)$$

$$= A^{s_{++} + 1} \langle D_{++} \rangle + A^{s_{+-} + 1} \langle D_{+-} \rangle + A^{-1} (A \langle D_{-+} \rangle + A^{-1} \langle D_{--} \rangle)$$

$$= A^{s_{1}} \langle s_1 D \rangle + A^{s_{2}} \langle s_2 D \rangle + \ldots + A^{s_{C}} \langle s_C D \rangle$$

$$= \sum_s A^{s} \left( -A^2 - A^{-2} \right)^{1_{s D_{1-1}}} = \sum_s \langle D_{1s} \rangle$$
Note: It is easy to see that the Kauffman bracket is well defined from this new definition.

**Def.** An achiral knot is one that is equivalent to its mirror image.

**Prop.** If \( K \) is achiral, then \( f_D(A) = f_D(A^{-1}) \).

**Pf.**

\[
\begin{align*}
& \quad A^{-1} \langle \alpha \rangle + A \langle \beta \rangle \\
\rightarrow & \quad A \langle \beta \rangle + A^{-1} \langle \alpha \rangle
\end{align*}
\]

equal when we substitute \( A^{-1} \) for \( A \).

Additionally, the rule

\[
\langle D \parallel U \rangle = (-A^2 - A^{-2}) \langle D \rangle
\]

is invariant under the substitution \( A \rightarrow A^{-1} \).

We now have a powerful way of showing knots are chiral. If \( f_D(A) \neq f_D(A^{-1}) \), then \( K \) is chiral (not achiral).
Ex) Calculate $f(A)$

$S_1 (1,1)$

$S_2 (1,-1)$

$S_3 (-1,1)$

$S_4 (-1,-1)$

$\sum S_1 = 2 \quad |S_1 D| = 2$

$\sum S_2 = 0 \quad |S_2 D| = 1$

$\sum S_3 = 0 \quad |S_3 D| = 1$

$\sum S_4 = -2 \quad |S_4 D| = 2$

$\langle D \rangle = A^2 (-A^2 - A^{-2})^1 + A^0 (-A^2 - A^{-2})^0$

$+ A^0 (-A^2 - A^{-2})^0 + A^{-2} (-A^2 - A^{-2})^1$

$= -A^4 - 1 + 1 + 1 - 1 - A^{-4}$

$= -A^4 - A^{-4}$

$f(A) = (-A)^{\frac{3}{2}} (-A^4 - A^{-4})$

$= -A^{-6} (-A^4 - A^{-4})$

$= A^{-2} + A^{-10}$
Knot Theory Day 11

Outline

- The Kauffman Bracket for alternating knots

**Def.** The breadth of a Laurent polynomial in $A$ is the difference between the highest and lowest powers in $A$ appearing.

**Ex.** $A^2 + 1 - A^{-3}$ has breadth $2 - (-3) = 5$.

Our Goal is to prove the following Theorem

**Thm.** The breadth of the Kauffman bracket of a reduced alternating knot diagram with $c$ crossings is exactly $4c$.

Corollaries

**Thm 2.**
The breadth of the Kauffman Bracket of any knot diagram is less than or equal to $4c$ where $c$ is the number of crossings.

1. Any reduced alternating knot diagram has minimal crossing number.
2. Any non-trivial reduced alternating knot diagram represents a non-trivial knot.
3. All reduced alternating diagrams of the same knot have the same crossing number.
Recall
- A state $s$ is an assignment of $\pm 1$ to all the crossings of a diagram $D$
- $SD$ is the resolution of $D$ according to the states.
- $|SD|$ is the number of loops in $SD$.
- $\sum s$ is the sum of the labels appearing in $s$.

Lemma 1: For a reduced alternating diagram
$|S_+D| > |S_-D|$ where $S_+$ is the state with all positive labels and $S_-$ is any state with exactly one negative label.

Proof: Let $D$ be an alternating reduced diagram.
The disks in the exterior of the knot projection in the plane of projection are called regions.

Label the regions red or blue according to the following rule at each crossing.

\[
\begin{array}{c}
R & | & B \\
\backslash & & / \\
B & | & R \\
\end{array}
\]
Lemma 2] Let $D$ be any knot diagram with $c$ crossings. Let $S_+ = S_0, S_1, \ldots, S_c = S$ be a chain of states s.t. $S_i$ has $i$ negative labels and $S_i$ differs from $S_{i+1}$ by exactly one label. Then the maximal power in $\langle D | S_{j+1} \rangle$ is less than or equal to the maximal power in $\langle D | S_j \rangle$.

**Proof** \( \langle D | S_j \rangle = A^{S_j} (-A^2 A^{-2})^{1_{S_j} D_1 - 1} \)
\[ = A^{c - 2j} (-A^2 A^{-2})^{1_{S_j} D_1 - 1} \]
\[ \langle D | S_{j+1} \rangle = A^{c - 2j - 2} (-A^2 A^{-2})^{1_{S_{j+1}} D_1 - 1} \]

However, $1_{S_j} D_1$ and $1_{S_{j+1}} D_1$ can differ by at most one.

So, highest power in $\langle D | S_j \rangle$ is $A^{c - 2j - 2} 1_{S_j} D_1 - 2$

$\langle D | S_{j+1} \rangle$ is at most $A^{c - 2j - 2 + 1_{S_j} D_1} \quad \Box$
The span of the Kauffmann bracket of a reduced alternating diagram with exactly \( c \) crossings is \( 4c \).

**Pf.** The highest power in \( \langle D | s_p \rangle = A^{s_{p+1}} (-A^2 - A^{-2})^{15, D1 - 1} \) is \( C + 21s_p D1 - 2 \).

Examine \( \langle D | s_1 \rangle \) where \( s_1 \) has exactly one minus label. \( \langle D | s_1 \rangle \) has highest power

\[
\langle D | s_1 \rangle = A^{s_{1+1}} (-A^2 - A^{-2})^{15, D1 - 1} = A^{C-2} (-A^2 - A^{-2})^{15, D1 - 1}
\]

\( \langle D | s_1 \rangle \) has highest power \( C - 2 + 21s_1 D1 - 2 \).

By lemma 1, \( 15, D1 + 1 = 15, D1 \).

So, \( \langle D | s_1 \rangle \) has highest power \( C - 2 + 21s_1 D1 - 2 - 2 \).

Thus, the highest power on \( \langle D | s_1 \rangle \) is four larger than that of \( \langle D | s_1 \rangle \). By Lemma 2, no other \( \langle D | s \rangle \) can contribute a term with highest power larger than the highest power of \( \langle D | s \rangle \).

By a symmetric argument, no \( \langle D | s \rangle \) contributes a term with lowest power as small as
the lowest power of $<D|s_->$.

$$<D|s_-> = A^s_- \left( -A^2 - A^{-2} \right)^{1-s_-D} - 1$$

$<D|s_->$ has lowest power $-c-2|s_-D|+2$.

Hence, the breadth of $<D>$ is

$$C+2|s_+D|-2 - (-c-2|s_-D|+2) = 2((s_+D)+s_-D)$$

However, it will be an easy Euler Characteristic calculation to show $|s_+D|+|s_-D| = C+2$.

Hence, the breadth of $<D>$ is $4c$. $\square$