Knot theory Day 10

Outline

- Jones Polynomial
- State sum formula for the Kauffman bracket
- Properties of the Jones Polynomial.

<u>Recall</u> Given a knot diagram D

$$f_D(A) = (-A)^{-3w(D)} \langle D \rangle \text{ is a knot invariant}$$

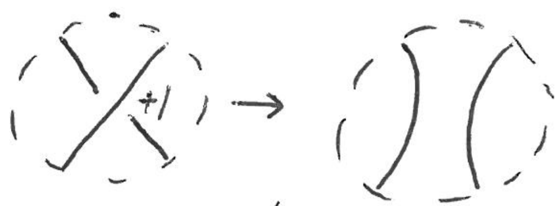
where $w(D)$ is the writhe and $\langle D \rangle$ is the Kauffman bracket

<u>Def</u> The Jones polynomial is $f_D(A)$ where we make the substitution $A = t^{-1/4}$.

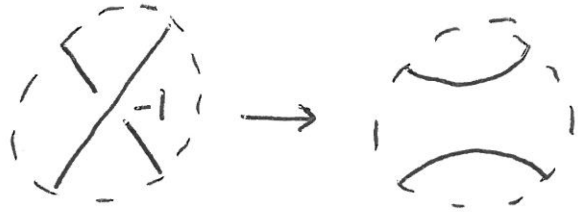
Von Jones won the fields medal for his work on the Jones poly. The Jones poly was a key tool use by others to solve all 3 Tait Conjectures.

<u>State Sum model for the Kauffman bracket</u>

Def A *state* s of a diagram D is an assignment of $+1$ or -1 to each crossing. (A diagram has $2^{c(D)}$ states). Given a state s on a diagram D, sD denotes a *resolution* of D into a crossingless diagram by the rule



and



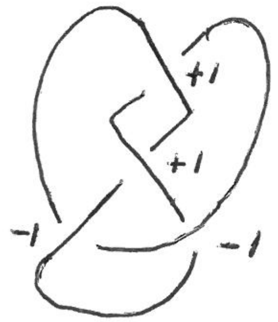
Let $|sD|$ denote the number of disjiont loops in sD.

Let Σs denote the sum of the values in a state s.

Remarks

1) Σs is between $-c$ and $+c$, but has the same parity as c.
2) If s and t are states for a diagram D that differ at exactly one crossing $|sD| = |tD| \pm 1$

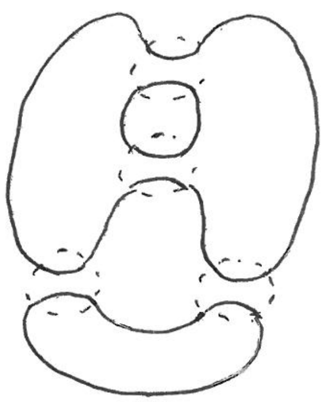
Ex

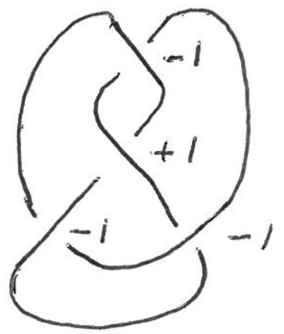


D

$\longrightarrow$

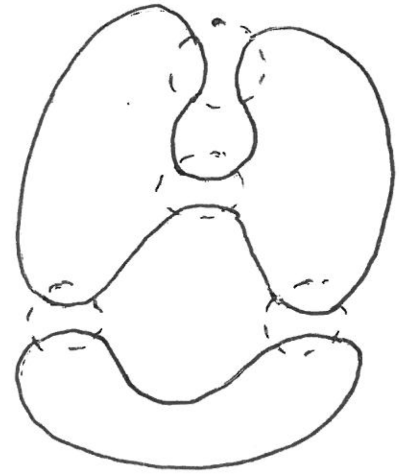
sD





D

$\longrightarrow$ tD



Prop

$$\langle D \rangle = \sum_s \langle D | s \rangle$$

where the sum is over all states of D and

$$\langle D | s \rangle = A^{\Sigma s} \left(-A^2 - A^{-2}\right)^{|s(D)|-1}$$

($\langle D|s\rangle$ is the "contribution" of the state s to the Kauffmann bracket)

Proof) - Let D be a diagram with c crossings.

- Order these crossings from 1 to c.
- Let D_+ denote the diagram D with crossing 1 resolved positively.
- Let D_{--} denote diagram D with crossing 1 resolved negatively and crossing 2 resolved positively ect.

$$\langle D \rangle = A \langle D_+ \rangle + A^{-1} \langle D_- \rangle$$

$$= A\left(A \langle D_{++} \rangle + A^{-1} \langle D_{+-} \rangle\right) +$$

$$A^{-1}\left(A \langle D_{-+} \rangle + A^{-1} \langle D_{--} \rangle\right.$$

$$= A^{+1+1} \langle D_{++} \rangle + A^{+1-1} \langle D_{+-} \rangle +$$

$$A^{-1+1} \langle D_{-+} \rangle + A^{-1-1} \langle D_{--} \rangle$$

$$= A^{\sum s_1} \langle s_1 D \rangle + A^{\sum s_2} \langle s_2 D \rangle + \cdots + A^{\sum s_{2^c}} \langle s_{2^c} D \rangle$$

$$= \sum_s A^{\sum s} \langle s D \rangle$$

$$= \sum_s A^{\sum s} (-A^2 - A^{-2})^{|sD|-1} = \sum_s \langle D | s \rangle \quad \square$$

Note: It is easy to see that the Kauffman bracket is well defined from this new deffinition.

Def An achiral knot is one that is equivalent to its mirror image

Prop If K is achiral, then $f_D(A) = f_D(A^{-1})$

Pf

$\downarrow \langle D \rangle \qquad\qquad \downarrow \langle \; \rangle$

$$A^{-1}\langle)(\rangle + A\langle \asymp \rangle \qquad A\langle)(\rangle + A^{-1}\langle \asymp \rangle$$

equal when we subsitute A^{-1} for A.

Additionally, the rule

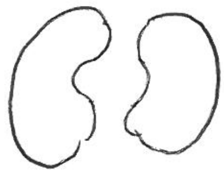
$$\langle D \amalg U \rangle = (-A^2 - A^{-2})\langle D \rangle$$

is invariant under the subsitution $A \to A^{-1}$ $\square$

We ~~kno~~ now have a powerful way of showing knots are chiral. If $f_D(A) \neq f_D(A^{-1})$, then K is chiral (not achiral).

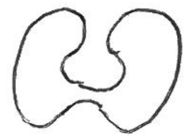
Ex Calculate f (A)

$S_1 \quad (+1,+1)$



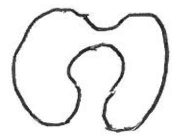
$\sum S_1 = 2 \qquad |S_1 D| = 2$

$S_2 \quad (+1,-1)$



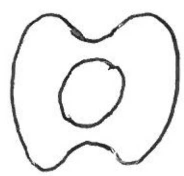
$\sum S_2 = 0 \qquad |S_2 D| = 1$

$S_3 \quad (-1,+1)$



$\sum S_3 = 0 \qquad |S_3 D| = 1$

$S_4 \quad (-1,-1)$



$\sum S_4 = -2 \qquad |S_4 D| = 2$

$$\langle D \rangle = A^2(-A^2-A^{-2})^1 + A^0(-A^2-A^{-2})^0 + A^0(-A^2-A^{-2})^0 + A^{-2}(-A^2-A^{-2})^1$$

$$= -A^4 - 1 + 1 + 1 - 1 - A^{-4}$$

$$= -A^4 - A^{-4}$$

$$f(A) = (-A)^{-3(2)}(-A^4 - A^{-4})$$

$$= -A^{-6}(-A^4 - A^{-4})$$

$$= A^{-2} + A^{-10}$$

Knot Theory Day 11

Outline

- The Kauffman Bracket for alternating knots

Def The breadth of a Laurent polynomial in A is the difference between the higest and lowest powers in A appearing.

Ex $A^2 + 1 - A^{-3}$ has breadth $2-(-3)=5$.

Our Goal is to prove the following Theorem

Thm 1 The breadth of the Kauffman bracket of a reduced alternating knot diagram with c crossings is exactly $4c$.

Thm 2 The breadth of the Kauffman Bracket of any knot diagram is less than or equal to $4c$ where c is the number of crossings.

Corollaries

1. Any reduced alternating knot diagram has minimal crossing number
2. Any non-trivial reduced alternating knot diagram represents a non trivial knot
3. All reduced alternating diagrams of the same knot have the same crossing number.

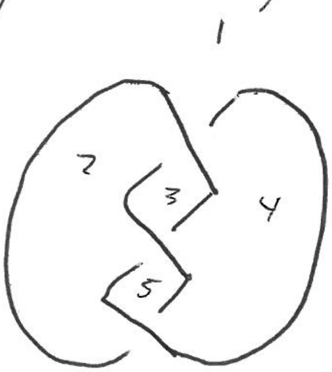
Recall

- A <u>state</u> s is an assignment of ± 1 to all the crossings of a diagram D
- sD is the resolution of D according to the state s.
- $|sD|$ is the number of loops in sD.
- $\sum s$ is the sum of the labels appearing in s.

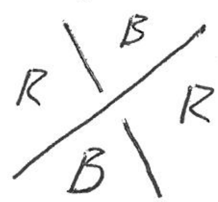
Lemma 1 For a reduced alternating diagram $|S_+ D| > |S_- D|$ where S_+ is the state with all positive labels and S_- is any state with exactly one negative lable.

Pf Let D be an alternating reduced diagram.

The disks in the exterior of the knot projection in the plane of projection are called regions.



Lable the regions red or blue according to the following rule at each crossing.



__Lemma 2__ Let D be any knot diagram with c crossings. Let $S_+ = S_0, S_1, \ldots, S_i = S$ be a chain of states s.t. S_i has i negative labels and S_i differs from S_{i+1} by exactly one label. Then the maximal power in $\langle D | s_{j+1} \rangle$ is less than or equal to the maximal power in $\langle D | s_j \rangle$.

__Pf__ Recall

$$\langle D | s_j \rangle = A^{\sum s_j}(-A^2 - A^{-2})^{|s_j D| - 1}$$
$$= A^{c-2j}(-A^2 - A^{-2})^{|s_j D| - 1}$$
$$\langle D | s_{j+1} \rangle = A^{c-2j-2}(-A - A^{-2})^{|s_{j+1} D| - 1}$$

However, $|s_j D|$ and $|s_{j+1} D|$ can differ by at most one.

So, higest power in $\langle D | s_j \rangle$ is $A^{c-2j+2|s_j D|-2}$

" " " $\langle D | s_{j+1} \rangle$ is at most $A^{c-2j-2+|s_j D|}$

$\square$

Thm | The span of the Kauffmann bracket of a reduced alternating diagram with exactly c crossings is $4c$.

Pf | The higest power in $\langle D|s_+\rangle = A^{\sum s_+}(-A^2-A^{-2})^{|s_+D|-1}$

is $c+2|s_+D|-2$.

Examine $\langle D|s_1\rangle$ where s_1 has exactly one minus label. ~~$\langle D|s_1\rangle$ has highest power~~

$$\langle D|s_1\rangle = A^{\sum s_1}(-A^2-A^{-2})^{|s_1D|-1}$$
$$= A^{c-2}(-A^2-A^{-2})^{|s_1D|-1}$$

$\langle D|s_1\rangle$ has highest power $c-2+2|s_1D|-2$.

By lemma 1, $|s_1D|+1=|s_+D|$.

So, $\langle D|s_1\rangle$ has higest power $c-2+2|s_+D|-2-2$.

Thus, the highest power on $\langle D|s_+\rangle$ is four larger than that of $\langle D|s_1\rangle$. By Lemma 2, no other $\langle D|s\rangle$ can contribute a term with highest power larger than the higest power of $\langle D|s_1\rangle$

By a symetric argument, no $\langle D|s\rangle$ contributes a term with lowest power as small as the lowest power of $\langle D|s_-\rangle$.

$$\langle D|s_-\rangle = A^{\sum s_-}\left(-A^{2}-A^{-2}\right)^{|s_-D|-1}$$

$\langle D|s_-\rangle$ has lowest power $-c-2|s_-D|+2$.

Hence, the breadth of $\langle D\rangle$ is

$$c+2|s_+D|-2-(-c-2|s_-D|+2) = 2c+(-4)+2\left(|s_+D|+|s_-D|\right)$$

However, it will be an easy Euler Charicteristic calculation to show $|s_+D|+|s_-D| = c+2$.

Hence, the breadth of $\langle D\rangle$ is $4c$. $\square$