Math 123: Power Series

Ryan Blair

CSU Long Beach

Thursday April 7, 2016
1 Power Series
Definition

A **Power Series** is a series and a function of the form

\[ P(x) = \sum_{k=0}^{\infty} c_k (x - a)^k = c_1 + c_2(x - a) + c_3(x - a)^2 + \ldots \]

where \( x \) is a variable, the \( c_i \) are constants and we say \( P(x) \) is centered at \( a \).

**For what values of \( x \) does a power series converge?**
Convergence of Power Series

A power series $\sum_{k=0}^{\infty} c_k (x - a)^k$ fits into one of the following three categories:

1. It converges for all $x$
2. It converges only at $x = a$
3. It converges for all $x$ such that $|x - a| < R$ where $R$ is some positive constant and may or may not converge at $x = a + R$ and $x = a - R$.

In the third case $R$ is called the **radius of convergence**.
Ratio Test

**Theorem**

Given a series \( \sum_{i=1}^{\infty} a_i \). If

\[
\lim_{i \to \infty} \left| \frac{a_{i+1}}{a_i} \right| = L,
\]

then

1. If \( L < 1 \), the series converges absolutely.
2. If \( L = 1 \), the test is inconclusive.
3. If \( L > 1 \), the series diverges.

**Example:** Use the ratio test to show \( \sum_{n=1}^{\infty} \frac{1}{n!} \) converges.
How to find the radius of convergence.

Using the ratio test we see that \( \sum_{k=0}^{\infty} c_k (x - a)^k \) converges if

\[
\lim_{k \to \infty} \left| \frac{c_{k+1}(x - a)}{c_k} \right| < 1
\]

So, we get

\[
R = \lim_{k \to \infty} \left| \frac{c_k}{c_{k+1}} \right|
\]
Power Series

How to find the radius of convergence.

Using the ratio test we see that $\sum_{k=0}^{\infty} c_k (x - a)^k$ converges if

$$\lim_{k \to \infty} \left| \frac{c_{k+1}(x - a)}{c_k} \right| < 1$$

So, we get

$$R = \lim_{k \to \infty} \left| \frac{c_k}{c_{k+1}} \right|$$

Find the radius of convergence of the power series for $\sum_{k=1}^{\infty} \frac{x^k}{k!}$. 
How to find the radius of convergence.

Using the ratio test we see that \( \sum_{k=0}^{\infty} c_k (x - a)^k \) converges if

\[
\lim_{k \to \infty} \left| \frac{c_{k+1}(x - a)}{c_k} \right| < 1
\]

So, we get

\[
R = \lim_{k \to \infty} \left| \frac{c_k}{c_{k+1}} \right|
\]

Find the radius of convergence of the power series for \( \sum_{k=1}^{\infty} \frac{x^k}{k!} \).

Find the radius of convergence for the power series \( \sum_{k=1}^{\infty} x^k \).
How to find the radius of convergence.

Using the ratio test we see that \( \sum_{k=0}^{\infty} c_k (x - a)^k \) converges if

\[
\lim_{k \to \infty} \left| \frac{c_{k+1}(x - a)}{c_k} \right| < 1
\]

So, we get

\[
R = \lim_{k \to \infty} \left| \frac{c_k}{c_{k+1}} \right|
\]

Find the radius of convergence of the power series for \( \sum_{k=1}^{\infty} \frac{x^k}{k!} \).
Find the radius of convergence for the power series \( \sum_{k=1}^{\infty} x^k \).
Find the radius of convergence for the power series \( \sum_{k=1}^{\infty} \frac{(3x)^k}{k5^k} \).
How to find the radius of convergence.

Using the ratio test we see that $\sum_{k=0}^{\infty} c_k (x - a)^k$ converges if

$$\lim_{k \to \infty} \left| \frac{c_{k+1}(x - a)}{c_k} \right| < 1$$

So, we get

$$R = \lim_{k \to \infty} \left| \frac{c_k}{c_{k+1}} \right|$$

Find the radius of convergence of the power series for $\sum_{k=1}^{\infty} \frac{x^k}{k!}$. Find the radius of convergence for the power series $\sum_{k=1}^{\infty} x^k$. Find the radius of convergence for the power series $\sum_{k=1}^{\infty} \frac{(3x)^k}{k5^k}$. If the radius of convergence of a power series is $R$, find the radius of convergence of its derivative.
Given a power series $\sum_{k=0}^{\infty} c_k (x - a)^k$ with radius of convergence $R$, the **interval of convergence** is one of the following where we include endpoints if the series is convergent at those points.

$$(a - R, a + R), [a - R, a + R), (a - R, a + R], [a - R, a + R]$$

Find the interval of convergence of $\sum_{k=1}^{\infty} \frac{(x)^k}{k}$.