Math 123: Taylor’s Formula and Approximations

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Taylor’s Formula

\[ f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x) \]

Where \( R_n(x) \) is the **error term of order** \( n \).
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Where \( R_n(x) \) is the error term of order \( n \).

Theorem (Taylor’s Theorem)

Given a Taylor Series \( \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \), if there is a constant \( M \) such that \( |f^{(n+1)}(t)| < M \) for all \( t \) between \( a \) and \( x \), then

\[ |R_n(x)| < M \frac{|x-a|^{n+1}}{(n+1)!} \]
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Uses: Can show Taylor series converges if \( |R_n(x)| \) goes to zero as \( n \) goes to infinity, Can get estimates for functions.
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Where \( R_n(x) \) is the **error term of order** \( n \).

**Theorem (Taylor’s Theorem)**

Given a Taylor Series \( \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \), if there is a constant \( M \) such that \( |f^{(n+1)}(t)| < M \) for all \( t \) between \( a \) and \( x \), then

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Uses: Can show Taylor series converges if \( |R_n(x)| \) goes to zero as \( n \) goes to infinity, Can get estimates for functions. Show that the Maclaurin series for \( \cos(x) \) converges to \( \cos(x) \) for all \( x \) using Taylor’s Theorem.
Examples

1. Show that the Maclaurin series for \( \frac{1}{1-x} \) converges to \( \frac{1}{1-x} \) for all \( x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \) by finding a formula for \( R_n(x) \).

2. Estimate the error for approximating \( e^x \) on \( \left[-\frac{1}{2}, \frac{1}{2}\right] \) using \( 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \).

3. Estimate the error for approximating \( \cos(x) \) on \( [-2\pi, 2\pi] \) using \( 1 + \frac{-x^2}{2} + \frac{x^4}{24} \).