Outline

1. Types of D.E.s

2. Solving D.E.s Using Auxiliary Equations
Types of Differential equations

**Definition**
A second order **linear** D.E. is of the form

\[ y'' + P(x)y' + Q(x)y = R(x) \]

If \( R(x) = 0 \) we call the D.E. **homogeneous**.

**Definition**
If \( P(x) \) and \( Q(x) \) are constants then \( y'' + P(x)y' + Q(x)y = R(x) \) is **constant coefficient**.
Theorem

Given a homogeneous linear differential equation with solutions $f(x)$ and $g(x)$ then $a \cdot f(x) + b \cdot g(x)$ is also a solution for any constants $a$ and $b$. 
Solutions to Homogeneous D.E.s

Theorem

Given a homogeneous linear differential equation with solutions \( f(x) \) and \( g(x) \) then \( a \cdot f(x) + b \cdot g(x) \) is also a solution for any constants \( a \) and \( b \).

Theorem

Given a 2nd order homogeneous linear differential equation with \textbf{linearly independent} solutions \( f(x) \) and \( g(x) \), then the general solution is \( y = C_1 f(x) + C_2 g(x) \) where \( C_1 \) and \( C_2 \) are constants.
A Motivating Example

Our goal is to solve constant-coefficient, linear, 2nd-order, homogeneous differential equations.
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In this case, we get $e^{mx}(am^2 + bm + c) = 0$. There are three possibilities for the roots of a quadratic equation.
Case 1: Distinct Roots

If $am^2 + bm + c$ has distinct roots $m_1$ and $m_2$, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$
Case 2: Repeated Roots

If $am^2 + bm + c$ has a repeated root $m_1$, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{m_1x} + c_2 xe^{m_1x}.$$
Solving D.E.s Using Auxiliary Equations

Magic!

\[ e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \ldots \]
Magic!

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\[ = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \ldots \]
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\[ = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots) \]
Magic!

\[ e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \ldots \]

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\[ = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots) \]

\[ = \cos(\theta) + isin(\theta) \]
Case 3: Complex Roots

If $am^2 + bm + c$ has complex roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$
Auxiliary Equations

Given a linear 2nd order homogeneous \textbf{constant-coefficient} differential equation

\[ ay'' + by' + cy = 0, \]

the \textbf{Auxiliary Equation} is

\[ am^2 + bm + c = 0. \]
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the **Auxiliary Equation** is

\[ am^2 + bm + c = 0. \]

The roots of the auxiliary equation determines the general solution.