Math 123: Trig Integrals and Trig Substitution

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Tuesday January 26, 2016
Outline

1. Trig Integrals

2. Trig Substitution
How do we integrate Trigonometric functions?

Example: \( \int \sin(x) \, dx \) and \( \int \cos(x) \, dx \).
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Example: \( \int \sin^3(x) \, dx \)
How do we integrate Trigonometric functions?

Example: \[ \int \sin(x) \, dx \text{ and } \int \cos(x) \, dx. \]
Example: \[ \int \sin^3(x) \, dx \]

For \( \int \cos^{odd}(x) \, dx \) or \( \int \sin^{odd}(x) \, dx \) use

\[ \cos^2(x) + \sin^2(x) = 1 \]

and u-substitution
How do we integrate Trigonometric functions?

Example: $\int \sin(x)\,dx$ and $\int \cos(x)\,dx$.

Example: $\int \sin^3(x)\,dx$

For $\int \cos^{\text{odd}}(x)\,dx$ or $\int \sin^{\text{odd}}(x)\,dx$ use

$$\cos^2(x) + \sin^2(x) = 1$$

and u-substitution

Example: $\int \cos^5(x)\,dx$
How do we integrate Trigonometric functions?

Example: \( \int \sin^2(x) \, dx \)
How do we integrate Trigonometric functions?

**Example:** \( \int \sin^2(x) \, dx \)

For \( \int \cos^{\text{even}}(x) \, dx \) or \( \int \sin^{\text{even}}(x) \, dx \) use

\[
\cos^2(x) = \frac{1}{2}(1 + \cos(2x))
\]

\[
\sin^2(x) = \frac{1}{2}(1 - \cos(2x))
\]

possibly multiple times
How do we integrate Trigonometric functions?

Example: \( \int \sin^2(x) \, dx \)

For \( \int \cos^{\text{even}}(x) \, dx \) or \( \int \sin^{\text{even}}(x) \, dx \) use

\[
\cos^2(x) = \frac{1}{2}(1 + \cos(2x))
\]

\[
\sin^2(x) = \frac{1}{2}(1 - \cos(2x))
\]

possibly multiple times

Example: \( \int \cos^4(x) \, dx \)
How do we integrate Trigonometric functions?

**Example:** \( \int \sin^2(x) \cos^3(x) \, dx \).
How do we integrate Trigonometric functions?

**Example:** \[ \int \sin^2(x)\cos^3(x)\,dx. \]

For \( \int \sin^{\text{anything}}(x)\cos^{\text{odd}}(x)\,dx \) or \( \int \cos^{\text{anything}}(x)\sin^{\text{odd}}(x)\,dx \) use

\[
\cos^2(x) + \sin^2(x) = 1
\]

and u-substitution
How do we integrate Trigonometric functions?

Example: \( \int \sin^2(x)\cos^3(x)\,dx \).

For \( \int \sin^{\text{anything}}(x)\cos^{\text{odd}}(x)\,dx \) or \( \int \cos^{\text{anything}}(x)\sin^{\text{odd}}(x)\,dx \) use

\[
\cos^2(x) + \sin^2(x) = 1
\]

and u-substitution

Question: What about \( \int \sin^{\text{even}}(x)\cos^{\text{even}}(x)\,dx \)
How do we integrate Trigonometric functions?

Example: \[ \int \tan(x) \sec^4(x) \, dx. \]
How do we integrate Trigonometric functions?

Example: $\int \tan(x)\sec^4(x)\,dx$.

For integrals involving $\tan(x)$ and $\sec(x)$ use

$$1 + \tan^2(x) = \sec^2(x)$$

and u-substitution.
Some Challenges

Example: Find $\int \sec(x) \, dx$.

Example: Find $\int \sec^3(x) \, dx$. 
A Motivating Example

Find $\int_{-1}^{1} \sqrt{1 - x^2} \, dx$ in two different ways.

**Method One:** Geometric.

**Method Two:** Using Trigonometric identities.
Find $\int_{-1}^{1} \sqrt{1 - x^2} \, dx$ in two different ways.

**Method One:** Geometric. Since $y = \sqrt{1 - x^2}$ is the top half of the unit circle, use definition of integral as area under the curve.

**Method Two:** Using Trigonometric identities.
A Motivating Example

Find $\int_{-1}^{1} \sqrt{1-x^2} \, dx$ in two different ways.

**Method One:** Geometric. Since $y = \sqrt{1-x^2}$ is the top half of the unit circle, use definition of integral as area under the curve.

**Method Two:** Using Trigonometric identities. Make the substitution $x = \sin(\theta)$ and use $\cos^2(\theta) + \sin^2(\theta) = 1$. 
Trig Substitution

For integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$ or $\sqrt{x^2 + a^2}$ where $a$ is a constant, we can often integrate by constructing a right triangle with one of these values as a side length.
Trig Substitution

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**Method Three:** Find $\int_{-1}^{1} \sqrt{1 - x^2} \, dx$ by building the relevant right triangle and making a substitution.
Trig Substitution

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**Method Three:** Find $\int_{-1}^{1} \sqrt{1 - x^2} \, dx$ by building the relevant right triangle and making a substitution.

**Example:** Find $\int \frac{1}{x^2\sqrt{x^2 + 9}} \, dx$

**Example:** Find $\int \frac{1}{\sqrt{x^2 - 4}} \, dx$