1. (10 pts) Solve the following D.E.

$$\frac{dy}{dx} = \frac{(x+1)tan(y)}{(x^2+1)sec^2(y)}$$

$$\int \frac{\sec^2(y)}{\tan(y)}\, dy = \int \frac{x+1}{x^2+1}\, dx$$

Let $u = \tan(y)$

$du = \sec^2(y)\, dy$

$$\int \frac{1}{u}\, du = \int \frac{x}{x^2+1}\, dx + \int \frac{1}{x^2+1}\, dx$$

$$\ln|u| = \int \frac{x}{x^2+1}\, dx + \tan^{-1}(x)$$

let $v = x^2+1$

$dv = 2x\, dx$

$$\ln|\tan(y)| = \frac{1}{2}\int \frac{1}{v}\, dv + \tan^{-1}(x)$$

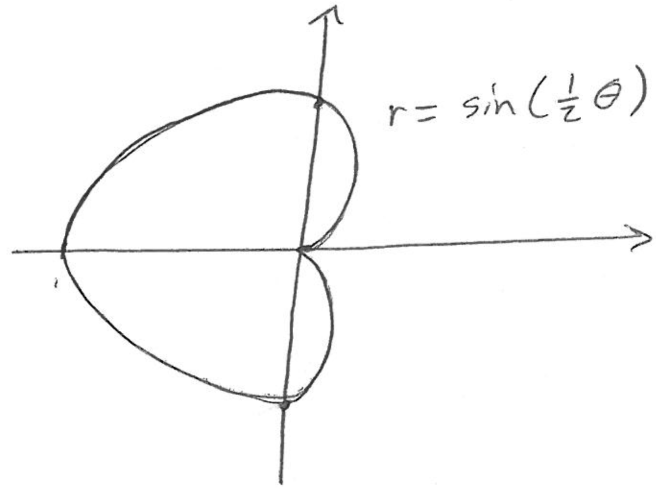
$$\ln|\tan(y)| = \frac{1}{2}\ln|v| + \tan^{-1}(x) + C$$

$$\tan(y) = \pm e^{\frac{1}{2}\ln|x^2+1| + \tan^{-1}(x) + C}$$

$$\boxed{y = \tan^{-1}\left(Ce^{\frac{1}{2}\ln|x^2+1| + \tan^{-1}(x)}\right)}$$

2. (10 pts) Sketch the graph of the polar equation $r = sin(\frac{1}{2}\theta)$ from $\theta = 0$ to $\theta = 2\pi$ and find the area enclosed by the curve.

θ	$r = \sin\left(\frac{\theta}{2}\right)$
0	0
$\frac{\pi}{2}$	$\frac{\sqrt{2}}{2}$
π	1
$\frac{3\pi}{2}$	$\frac{\sqrt{2}}{2}$
π	0



$$\text{Area} = \int_0^{2\pi} \frac{1}{2}\left(\sin\left(\tfrac{1}{2}\theta\right)\right)^2 d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\sin^2\left(\tfrac{1}{2}\theta\right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\sin\theta\right) d\theta$$

$$= \frac{1}{4}\theta + \frac{1}{4}\cos\theta \Big|_0^{2\pi}$$

$$= \frac{\pi}{2} + \frac{1}{4} - \left(0 + \frac{1}{4}\right) = \boxed{\frac{\pi}{2}}$$

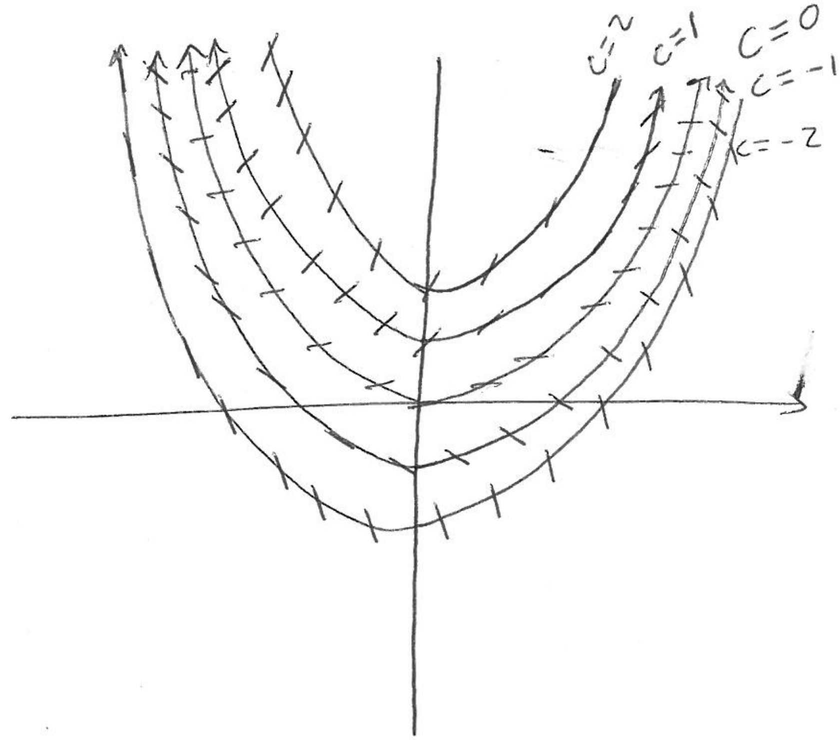
3. (10 pts)

A) Find the isoclines for $y' = y - x^2$ and use them to graph the slope field.

Isoclines

$C = y - x^2$

$y = x^2 + C$



B) Use the slope field to determine for what values of b does the IVP $y(0) = b$ and $y' = y - x^2$ have a solution that is strictly decreasing.

$b \leq 0$

C) Find $lim_{x \to -\infty} f(x)$ for any solution $f(x)$ to $y' = y - x^2$.

$$\lim_{x \to -\infty} f(x) = +\infty$$

4. (10 pts) Find all points with horizontal and vertical tangents on the polar curve $r = e^\theta$

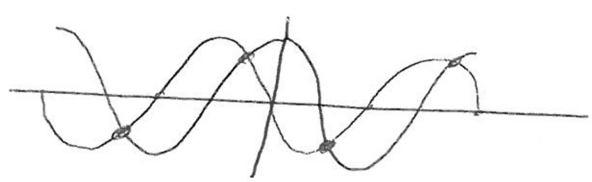
$$r = e^\theta \Rightarrow \begin{array}{l} x = e^\theta \cos\theta \\ y = e^\theta \sin\theta \end{array}$$

$$\frac{dx}{d\theta} = -e^\theta \sin\theta + e^\theta \cos\theta \qquad \frac{dy}{d\theta} = e^\theta \cos\theta + e^\theta \sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{e^\theta \cos\theta + e^\theta \sin\theta}{-e^\theta \sin\theta + e^\theta \cos\theta} = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$$

horz. ~~vert.~~ tangents

$$0 = \cos\theta + \sin\theta$$
$$\cos\theta = -\sin\theta$$
$$\theta = -\frac{\pi}{4} + n\pi \quad n \in \mathbb{Z}$$

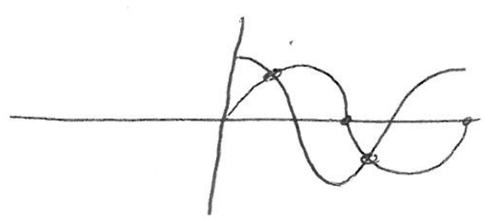


All points $(x, y) = \left(e^{-\frac{\pi}{4} + n\pi} \cos\left(-\frac{\pi}{4} + n\pi\right), e^{-\frac{\pi}{4} + n\pi} \sin\left(-\frac{\pi}{4} + n\pi\right)\right)$

$n \in \mathbb{Z}$

Vert. tangents

$$0 = \cos\theta - \sin\theta$$
$$\cos\theta = \sin\theta$$
$$\theta = \frac{\pi}{4} + n\pi \quad n \in \mathbb{Z}$$



All points $(x, y) = e^{\frac{\pi}{4} + n\pi} \cos\left(\frac{\pi}{4} + n\pi\right), e^{\frac{\pi}{4} + n\pi} \sin\left(\frac{\pi}{4} + n\pi\right))$

5. Derive Euler's formula using the Taylor series for e^x.

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots$$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

6. (10 pts) Show that if $b \neq 0$, $e^{ax}cos(bx)$ and $e^{ax}sin(bx)$ are linearly independent functions.

Suppose, to form a contradiction, that $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$ are L.D.

There is a constant C s.t.

$$e^{ax}\cos(bx) = c\,e^{ax}\sin(bx)$$

Let $x = 0$.

$$e^{0}\cos(0) = c\,e^{0}\sin(0)$$

$$1 \cdot 1 = c \cdot 1 \cdot 0$$

$$1 = 0 \leftarrow \text{this is a contradiction.}$$

Hence $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$ are L.I.

Typo!

7. (10 pts) Solve $y'' - 3y = 0$.

This is a linear, homogeneous, 2nd-order constant coefficient D.E. (LH2CDE).

Aux. Eq. $m^2 - 3 = 0$

$$m^2 = 3$$

$$m = \pm\sqrt{3}$$

Distinct real roots

$$\boxed{y = c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x}}$$

8. (10 pts) Solve $y'' + 2y' + y = 0$.

This is a LH2CDE.

Aux. Eq. $m^2 + 2m + 1 = 0$

$(m+1)(m+1) = 0$

$m = -1$

Repeated real root

$$\boxed{y = c_1 e^{-x} + c_2 x e^{-x}}$$

9. (10 pts) Solve $y'' + 2y' + 2y = 0$.

This is a LH2CDE

Aux. Eq. $m^2 + 2m + 2 = 0$

$$m = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$m = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$$

Complex roots

$$\boxed{y = c_1 e^{-x} \cos(x) + c_2 e^{-x} \sin(x)}$$

10. (10 pts) Solve $xy' + \frac{y}{ln(x)} = x^2$.

$$y' + \frac{y}{x\ln(x)} = x$$

Use ~~the~~ integrating factor method

$$e^{\int P(x)\,dx} = e^{\int \frac{1}{x\ln(x)}dx} = e^{\int \frac{1}{u}du} = e^{\ln|u|} = |u| = \ln(x)$$

let $u = \ln(x)$

Multiply both sides by the integrating factor

$$\ln(x)\left(y' + \frac{y}{x\ln(x)}\right) = \ln(x)x$$

$$\frac{d}{dx}\left(\ln(x)\,y\right) = x\ln(x)$$

$$\ln(x)\,y = \int x\ln(x)\,dx$$

use by parts

$u = \ln(x)$ $\quad v' = x$

$du = \frac{1}{x}$ $\quad v = \frac{x^2}{2}$

$$\ln(x)\,y = \frac{x^2}{2}\ln(x) - \int \frac{x}{2}dx = \frac{x^2}{2}\ln(x) - \frac{x^2}{4} + C$$

$$\boxed{y = \frac{\frac{x^2}{2}\ln(x) - \frac{x^2}{4} + C}{\ln(x)}}$$