Sequences

- L'Hospitals √
- Squeeze theorem √
- Monotone & bounded √

Series

- test for divergence √
- Comparison test √
- limit comparison test √
- integral test √
- root test √
- ratio test √
- Alternating Series test √
- Absolut and conditional convergence √

Power Series

- Find Power Series √
- Interval of convergence of power Series
- Taylor Series
- Taylor's formula for approximation

Power Series

A <u>power series</u> is a function $P(x)$ of the form $P(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$.

use $\frac{1}{1-x} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

Ex| Find the power series for $\tan^{-1}(x)$

Find the power series for $f''(x)$, if $f(x) = \frac{1}{1+x}$.

Def: Given $P(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$ the <u>radius of convergence</u>

$$R = \lim_{n\to\infty} \left| \frac{C_n}{C_{n+1}} \right|$$

Ex| Use the ratio test to derive the ~~radius of~~ formula for the radius of convergence.

Ex| Find the radius of convergence for $\sum_{n=0}^{\infty} \frac{(100x)^n}{n!}$

Def: The interval of convergence for $P(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$ is $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R]$ where we include endpoints if the series converges at those endpoints.

Ex] Find the interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$$

Ex] Find the interval of convergence for

$$\sum_{n=1}^{\infty} n^n x^n$$

— Taylor Series

Given a function $f(x)$ the taylor Series for $f(x)$ at $x=a$ is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

* Just need to find the pattern for $f^{(n)}(a)$. *

Ex] Find the T.S. for $\cos(x)$ at $x=0$

Ex] Find the T.S. for $\ln(1+2x)$ $\cancel{\frac{1}{1+2x}}$ at $x=0$.

— Taylor's Formula

Let $R_n(x)$ be the error in approximating $f(x)$ by the first $n+1$ terms of the Taylor Series for $f(x)$ at $x=a$.

$$|R_n(x)| \le \frac{M\,|x-a|^{n+1}}{n+1} \quad \text{for } x \in (c,d)$$

where $|f^{(n+1)}(t)| \le M$ for $t \in (c,d)$

Ex/ Find a bound for $R_3(x)$ on $[1, 3]$ where $f(x) = \ln(x)$ and we use the Taylor Series centered at $x = 2$.

Ex/ Find a bound for $R_2(x)$ on $[-1, 1]$ where $f(x) = \int e^{x^2} dx$ using the taylor Series centered at $x=0$.

Sequences

Known limits $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$

$\lim_{n\to\infty} n^{1/n} = 1$

L'Hospital's rule is the main tool to use when finding limits of Sequences

Th^m (Squeeze Th^m)

If $\{a_n\}$ ~~and~~, $\{b_n\}$ and $\{c_n\}$ are sequences s.t.

$a_n \le b_n \le c_n$ for all ~~an~~ n and

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then

$\lim_{n\to\infty} b_n = L$.

Uses $\lim_{n\to\infty} \frac{\sin(n)}{n}$, $\lim_{n\to\infty} \frac{5^n}{n!}$

$\frac{-1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}$ $\quad$ $0 \le \frac{5^n}{n!} \le \frac{5}{n}(1)(1)\cdots(\frac{5}{4})(\frac{5}{3})(\frac{5}{2})(\frac{5}{1})$

Th^m If $\{a_n\}$ is bounded above (bellow) and increasing (decreasing), then $\lim_{n\to\infty} a_n$ exists.

use on recursively defined sequences

$a_n = \sqrt{3 + a_{n-1}}$, $a_1 = \sqrt{3}$ $\quad$ or $\quad$ $a_1 = 2$ $\quad a_{n+1} = \frac{1}{3-a_n}$

Suppose $\lim_{n\to\infty} a_n = L$, to find it plug in to recursion relation

$$a_n = \sqrt{3 + a_{n-1}} = \quad L = \sqrt{3+L}$$

Often useful to use induction to show bounded or monotone

Series

- Test for divergence

  Given a Series $\sum a_n$, if $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ is divergent.

  Good test to use first

  $\sum \frac{n}{\ln(n)}$ diverges $\qquad \sum n^2$ diverges

- Comparison test.

  Given ~~If~~ $\{a_n\}$ and $\{b_n\}$ are Series s.t. $0 \le a_n \le b_n$ for all n

  If $\sum b_n$ converges, then $\sum a_n$ converges

  If $\sum a_n$ diverges, then $\sum b_n$ diverges

  $\sum \frac{|\sin(n)|}{n^2}$ converges, $\qquad \sum \frac{\ln(n)}{n}$ diverges

  compare to $\frac{1}{n^2}$ $\qquad$ compare to $\frac{1}{n}$

- Limit Comparison test

If $\sum a_n$ and $\sum b_n$ have positive terms s.t.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = C \neq 0 \text{ then}$$

either both $\sum a_n$ and $\sum b_n$ diverge or both converge.

$\sum \frac{n+1}{n^2+1}$ diverges

compare to $\frac{1}{n}$

$\sum \frac{e^n}{e^{2n}+n}$ converges

compare to $\sum \frac{1}{e^n}$

- Integral test

Given a positive, decreasing function $f(x)$ s.t. $a_n = f(n)$.

① If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum a_n$ is convergent

② If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum a_n$ is divergent

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$\sum n e^{-n}$ is convergent

$\sum \frac{\ln(\ln(n))}{n \ln(n)}$ is divergent.

Ratio test:

Given a ~~seq~~ series $\sum a_n$ s.t.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \text{, then}$$

if $L<1$, $\sum a_n$ is absolutly convergent

if $L>1$, $\sum a_n$ is divergent

if $L=1$, inconclusive.

Ex] $\sum_{n=1}^{\infty}\frac{n^4}{4^n}$

Ex] $\sum_{n=1}^{\infty}\frac{n^n}{n!}$

- use when we see - constants to powers
  - factorials
  - iterated multiplication $3\cdot5\cdot7\cdot9\cdots(2n+1)$

Root test

Given a series $\sum a_n$ s.t.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = L$$

if $L<1$, $\sum a_n$ is absolutely convergent

if $L>1$, $\sum a_n$ is divergent

if $L=1$, inconclusive.

Ex] $\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$

Ex] $\sum_{n=1}^{\infty}\left(1+\frac{2}{n}\right)^{n^2}$

- use when we see variables as powers.

Alternating Series test

Given $\sum a_n = \sum (-1)^n b_n$ s.t. $b_n \geq 0$

If $\lim_{n\to\infty} b_n = 0$ and $b_{n+1} \leq b_n$ for all n, then

$\sum a_n$ converges.

<u>Ex</u>| $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$

<u>Ex</u>| $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(\pi n)\, n^{1/2}}{n + n^{1/3}}$

- use when you suspect something converges conditionally
- $(-1)^n$ or $\cos(\pi n)$

- A sequence $\sum a_n$ is absolutely convergent if $\sum |a_n|$ is convergent
- A sequence $\sum a_n$ is conditionally convergent if $\sum |a_n|$ diverges and $\sum a_n$ converges.

<u>Ex</u>| Show $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ is conditionally convergent

Show $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n) + \sin(2n) + \sin(3n)}{n^2}$ is absolutely convergent.