Math 123: Taylor and Maclaurin Series

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Outline

1. Taylor Series
Taylor Series

Definition

The **Taylor series** generated by a function \( f \) at \( x = a \) is

\[
\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \ldots
\]
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Theorem

*If* $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k$ *has radius of convergence $R$, then*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k = f(x)$$

*for all $x$ in $(a - R, a + R)$*
Taylor Series are closely related to approximations

**Example:** Graph the following functions side-by-side:

- $e^x$
- $1$
- $1 + x$
- $1 + x + \frac{x^2}{2}$
- $1 + x + \frac{x^2}{2} + \frac{x^3}{6}$
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**Core Idea:** A Taylor Series is the LIMIT of successively better polynomial approximations!
Tricks to finding Taylor Series

**Problem:** Find the Taylor series for \( f(x) = \ln(x + 1) \) at \( x = 0 \).

**Trick:** No trick, just substitute into the formula for Taylor series and find the pattern.
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Answer: \( \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} \)
**Problem:** Find the Taylor series for $f(x) = \ln(x)$ at $x = 1$.

**Trick:** Save yourself time and use the Taylor Series we just found.
Tricks to finding Taylor Series

**Problem:** Find the Taylor series for \( f(x) = \ln(x) \) at \( x = 1 \).

**Trick:** Save yourself time and use the Taylor Series we just found.

**Answer:** \( \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(x-1)^k}{k} \)
Tricks to finding Taylor Series

**Problem:** Find the first 3 terms of the Taylor series for \( f(x) = x\sin(3x) \) at \( x = 0 \).

**Trick:** Use the fact that you know that Taylor Series for \( \sin(x) \).
Tricks to finding Taylor Series

Problem: Find the first 3 terms of the Taylor series for $f(x) = e^x \sin(x)$ at $x = 0$.

Trick: Use the fact that you know that Taylor Series for $\sin(x)$ and you know the Taylor Series for $e^x$. 