Math 123: Sequences Part II and Introduction to Series

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Outline

Convergence and Divergence

Definition

A **sequence** is an ordered set of real numbers, equivalently, a **sequence** is an function from the positive integers to the real numbers.

If $lim_{n \to \infty} a_n$ does not exist or is infinite we say it **diverges**.

Examples of sequences that diverge

$$a_n = (-1)^n$$

$$a_n = 2^n$$

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Exercise: If $r \in \mathbb{R}$, when does $a_n = r^n$ converge and diverge? (this is called a geometric sequence)

Alternating Sequences

An **alternating** sequence is of the form $a_n = (-1)^n b_n$ where $b_n \geq 0$ for all n.

Theorem

Given an alternating sequence a_n, if $lim_{n\to\infty}|a_n| = 0$ then $lim_{n\to\infty} a_n = 0$.

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Exercise: Prove the above theorem using our limit rules and the squeeze theorem.

Monotonic Sequences

Definition

A sequence is **increasing** if $a_n \leq a_{n+1}$ for all n.
A sequence is **decreasing** if $a_n \geq a_{n+1}$ for all n.
If a sequence is decreasing or increasing we say it is **monotonic**.

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A sequence is **bounded above** if there exists a constant M such that $a_n \leq M$ for all n.
A sequence is **bounded below** if there exists a constant m such that $a_n \geq m$ for all n.
A sequence is **bounded** if it is both bounded above and bounded below.

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Example: Suppose $a_1 = \sqrt{2}$ and $a_n = \sqrt{2 + a_{n-1}}$, show that $\{a_n\}$ converges and find its limit
Example: Suppose $a_1 = 1$ and $a_n = 3 - \frac{1}{a_{n-1}}$, show that $\{a_n\}$ converges and find its limit

Must Know Theorems Regarding Limits

1. $lim_{n\to\infty}\frac{ln(n)}{n} = 0$
2. $lim_{n\to\infty}n^{\frac{1}{n}} = 1$
3. $lim_{n\to\infty}x^{\frac{1}{n}} = 1$ if $x > 0$
4. $lim_{n\to\infty}x^{n} = 0$ if $|x| < 1$
5. $lim_{n\to\infty}(1+\frac{x}{n})^{n} = e^{x}$
6. $lim_{n\to\infty}\frac{x^n}{n!} = 0$

Series in terms of Sequences

Roughly, an infinite series $\sum_{i=1}^{\infty} a_i$ denotes the sum of the terms in the sequence $\{a_i\}_{i=1}^{\infty}$.

Definition

The **n-th partial sum** for a sequence $\{a_i\}_{i=1}^{\infty}$ is

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + ... + a_n$$

Definition

A **Series**

$$\sum_{i=1}^{\infty} a_i = lim_{n \to \infty} S_n$$

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Exercise: Given a constant r find $\Sigma_{i=0}^{\infty} r^i$ when it exists.

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Exercise: Given a constant r find $\Sigma_{i=0}^{\infty} r^i$ when it exists.
Exercise: Use partial sums to find $\Sigma_{i=1}^{\infty} \frac{1}{i^2+i}$.

First tests for convergence

Theorem

If a series $\sum_{i=1}^{\infty} a_i$ *converges then* $lim_{n\to\infty} a_i = 0$.

Theorem

If $lim_{n\to\infty} a_i \neq 0$ *or does not exist, then* $\sum_{i=1}^{\infty} a_i$ *does not converge.*

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If $lim_{n\to\infty} a_i \neq 0$ or does not exist, then $\Sigma_{i=1}^{\infty} a_i$ does not converge.

Exercise: Determine the convergence or divergence of $\Sigma_{i=1}^{\infty} ln(\frac{i^2+1}{2i^2+1})$

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Exercise: Determine the convergence or divergence of $\sum_{i=1}^{\infty} \frac{e^i}{i^2}$.