Math 123: Sequences Part II and Introduction to Series

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Outline

1. Sequences

2. Series
Convergence and Divergence

**Definition**

A **sequence** is an ordered set of real numbers, equivalently, a sequence is a function from the positive integers to the real numbers.

If \( \lim_{n \to \infty} a_n \) does not exist or is infinite we say it **diverges**.

Examples of sequences that diverge

\[ a_n = (-1)^n \]

\[ a_n = 2^n \]
Convergence and Divergence

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Exercise: If \( r \in \mathbb{R} \), when does \( a_n = r^n \) converge and diverge? (this is called a geometric sequence)
An alternating sequence is of the form $a_n = (-1)^n b_n$ where $b_n \geq 0$ for all $n$.

**Theorem**

Given an alternating sequence $a_n$, if $\lim_{n \to \infty} |a_n| = 0$ then $\lim_{n \to \infty} a_n = 0$. 

**Exercise:** Prove the above theorem using our limit rules and the squeeze theorem.
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Monotonic Sequences

**Definition**

A sequence is **increasing** if $a_n \leq a_{n+1}$ for all $n$.  
A sequence is **decreasing** if $a_n \geq a_{n+1}$ for all $n$.  
If a sequence is decreasing or increasing we say it is **monotonic**.
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A sequence is **increasing** if \( a_n \leq a_{n+1} \) for all \( n \).

A sequence is **decreasing** if \( a_n \geq a_{n+1} \) for all \( n \).

If a sequence is decreasing or increasing we say it is **monotonic**.

Definition

A sequence is **bounded above** if there exists a constant \( M \) such that \( a_n \leq M \) for all \( n \).

A sequence is **bounded below** if there exists a constant \( m \) such that \( a_n \geq m \) for all \( n \).

A sequence is **bounded** if it is both bounded above and bounded below.
Monotonic Sequences

Theorem

Every increasing sequence that is bounded above converges. Similarly, every decreasing sequence that is bounded below converges.
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Theorem

Every increasing sequence that is bounded above converges. Similarly, every decreasing sequence that is bounded below converges.

Example: Suppose $a_1 = \sqrt{2}$ and $a_n = \sqrt{2 + a_{n-1}}$, show that $\{a_n\}$ converges and find its limit.
Monotonic Sequences

**Theorem**

*Every increasing sequence that is bounded above converges. Similarly, every decreasing sequence that is bounded below converges.*

**Example:** Suppose \( a_1 = \sqrt{2} \) and \( a_n = \sqrt{2 + a_{n-1}} \), show that \( \{a_n\} \) converges and find its limit

**Example:** Suppose \( a_1 = 1 \) and \( a_n = 3 - \frac{1}{a_{n-1}} \), show that \( \{a_n\} \) converges and find its limit
Must Know Theorems Regarding Limits

1. \( \lim_{n \to \infty} \frac{\ln(n)}{n} = 0 \)
2. \( \lim_{n \to \infty} n^{\frac{1}{n}} = 1 \)
3. \( \lim_{n \to \infty} x^{\frac{1}{n}} = 1 \) if \( x > 0 \)
4. \( \lim_{n \to \infty} x^n = 0 \) if \( |x| < 1 \)
5. \( \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x \)
6. \( \lim_{n \to \infty} \frac{x^n}{n!} = 0 \)
Series in terms of Sequences

Roughly, an infinite series $\sum_{i=1}^{\infty} a_i$ denotes the sum of the terms in the sequence $\{a_i\}_{i=1}^{\infty}$.

Definition

The **n-th partial sum** for a sequence $\{a_i\}_{i=1}^{\infty}$ is

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

Definition

A **Series**

$$\sum_{i=1}^{\infty} a_i = \lim_{n \to \infty} S_n$$
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**Exercise**: Given a constant $r$ find $\sum_{i=0}^{\infty} r^i$ when it exists.
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**Exercise:** Given a constant $r$ find $\sum_{i=0}^{\infty} r^i$ when it exists.

**Exercise:** Use partial sums to find $\sum_{i=1}^{\infty} \frac{1}{i^2+i}$. 
First tests for convergence

**Theorem**

*If a series $\sum_{i=1}^{\infty} a_i$ converges then $\lim_{n \to \infty} a_i = 0$.***

**Theorem**

*If $\lim_{n \to \infty} a_i \neq 0$ or does not exist, then $\sum_{i=1}^{\infty} a_i$ does not converge.*
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**Exercise:** Determine the convergence or divergence of \( \sum_{i=1}^{\infty} \ln\left(\frac{i^2+1}{2i^2+1}\right) \)
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**Exercise:** Determine the convergence or divergence of \( \sum_{i=1}^{\infty} \frac{e^i}{i^2} \).