u-sub:

$$\boxed{\int f'(u)\,du = f(u) + C}$$

Use on

$\int \frac{1}{x\ln(x)}\,dx$ $\quad$ $\int \cos(x)\sin^2(x)\,dx$ $\quad$ $\int \frac{\tan^{-1}(x)}{x^2+1}\,dx$

$u = \ln(x)$ $\quad$ $u = \sin^2(x)$ $\quad$ $u = \tan^{-1}(x)$

Ex $\int \tan^2(x)\sec^2(x)\,dx$ $\quad$ $u = \tan(x)$, $du = \sec^2(x)$

$\int u^2\,du$

$\frac{u^3}{3} + C = \boxed{\frac{1}{3}\tan^3(x) + C}$

Integration by parts

$$\boxed{\int u v'\,dx = uv - \int u' v\,dx}$$

use on

$\int x e^x$ $\quad$ $\int \cos(x) e^x$ $\quad$ $\int x\ln(x)\,dx$ $\quad$ $\int \ln(x)\,dx$ $\quad$ $\int \tan^{-1}(x)\,dx$

$u = x$ $\quad$ $u = \cos(x)$ $\quad$ $u = \ln(x)$ $\quad$ $u = \ln(x)$ $\quad$ $u = \tan^{-1}(x)$

Ex $\int \cos(x) e^x\,dx$ $\quad$ $u = \cos(x)$ $\quad v' = e^x$

$u' = -\sin(x)$ $\quad v = e^x$

$= \cos(x)e^x + \int \sin(x) e^x\,dx$ $\quad$ $u = \sin(x)$ $\quad v' = e^x$

$u' = \cos(x)$ $\quad v = e^x$

$= \cos(x)e^x + e^x\sin(x) - \int e^x\cos(x)\,dx$

$\frac{1}{2}(\cos(x)e^x + e^x\sin(x)) + C$

Partial fractions

use on most $\displaystyle\int \frac{\text{Poly}}{\text{Poly}}\,dx$

$$\boxed{\text{Careful} \quad \int \frac{x}{x^2+1}\,dx \leftarrow \begin{array}{l} u\text{-sub} \\ u = x^2+1. \end{array}}$$

Ex| ~~Since deg~~ $\displaystyle\int \frac{x^2}{x^2-1}\,dx$

Since deg (Top) $\geq$ deg (bottom) use long division

$$x^2 \overline{\big)\, x^2 + 0x - 1} \quad 1 + \frac{-1}{x^2-1}$$
$$-x^2$$
$$-1$$

$$\int \frac{x^2}{x^2-1}\,dx = \int 1 + \frac{-1}{x^2-1}\,dx = \int 1 + \frac{-1/2}{x-1} + \frac{1/2}{x+1}\,dx$$

$$= x - \tfrac{1}{2}\ln|x-1| + \tfrac{1}{2}\ln|x+1| + C$$

Find A and B s.t.

$$\frac{A}{x+1} + \frac{B}{x-1} = \frac{-1}{(x+1)(x-1)}$$

$$A(x-1) + B(x+1) = -1$$

if $x = 1$ $\quad 2B = -1 \quad B = -1/2$

if $x = -1$ $\quad (-2)A = -1 \quad A = 1/2$

what do we guess for

$$\frac{1}{(x-2)(x-3)^2(x^2+1)} = \boxed{\frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{Dx+E}{x^2+1}}$$

Trig sub

Careful $\int \frac{x}{\sqrt{1+x^2}}$ u-sub $u = 1+x^2$

use when you see $\sqrt{\text{constant} \pm x^2}$

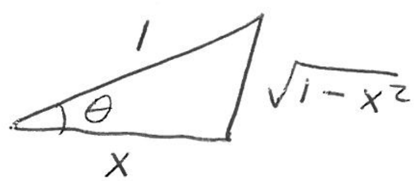
Use on

$$\int \sqrt{1-x^2}\, dx \qquad \int \frac{1}{(1+x^2)^{3/2}}\, dx$$

Ex

$$\int \frac{1}{x\sqrt{1-x^2}}\, dx = \int \frac{1}{\cos\theta \sin\theta} (-\sin\theta\, d\theta)$$

Make the triangle



$\cos\theta = x$

$-\sin\theta\, d\theta = dx$

$\sin\theta = \sqrt{1-x^2}$

$$= -\int \sec\theta\, d\theta$$

$$= -\int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}\, d\theta$$

$$= -\int \frac{1}{u}\, du$$

$u = \sec\theta + \tan\theta$

$du = \sec^2\theta + \tan\theta\sec\theta\, d\theta$

$$= -\ln|u| + C$$

$$= -\ln|\sec\theta + \tan\theta| + C$$

$$= \boxed{-\ln\left|\frac{1}{x} + \frac{\sqrt{1+x^2}}{x}\right| + C}$$

Trig integrals

$$\int \cos^n(x) \cdot \sin^m(x)\,dx$$

use u-sub or double angle formula.
with $\sin^2(x) + \cos^2(x) = 1$.

$$\int \sec^n(x) \cdot \tan^m(x)$$

- break into sin and cos
- use u sub
- $1 + \tan^2(x) = \sec^2(x)$.

- Improper integrals

$$\int_a^\infty f(x)\,dx \qquad \int_{-\infty}^b f(x)\,dx \qquad \int_{-\infty}^\infty f(x)\,dx \qquad \int_a^b f(x)\,dx.$$

where f is discontinuou on $[a,b]$.

use on

$$\int_0^\infty \frac{\tan^{-1}(x)}{x^2+1}\,dx \qquad \int_0^\infty e^{-x}\,dx \qquad \int_0^1 \frac{1}{x^{2/3}}\,dx$$

Ex|

$$\int_0^\infty \frac{\tan^{-1}(x)}{x^2+1}\,dx = \lim_{a\to\infty} \int_0^a \frac{\tan^{-1}(x)}{x^2+1}\,dx$$

let $u = \tan^{-1}(x)$
$du = \frac{1}{x^2+1}\,dx$

$$= \lim_{a\to\infty} \int_?^? u\,du$$

$$= \lim_{a\to\infty} \frac{u^2}{2}\Big|_?^? = \lim_{a\to\infty} \frac{1}{2}\left(\tan^{-1}(x)\right)^2\Big|_0^\infty = \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$$

~~If~~ Given $0 \le f(x) \le g(x)$

if $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges

if $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges

Ex | Show $\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx$ converges

since $0 \le \frac{\sin^2(x)}{x^2} \le \frac{1}{x^2}$

Look at $\int_\phi^\infty \frac{1}{x^2}\,dx = \lim_{a\to\infty} \int_\phi^a \frac{1}{x^2}\,dx = \lim_{a\to\infty} \cancel{\ln|a| - \ln|1|} \; -\frac{1}{a} + \frac{1}{1} = 1$

$\cancel{= \infty}$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges, then $\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx$ converges.

Volumes

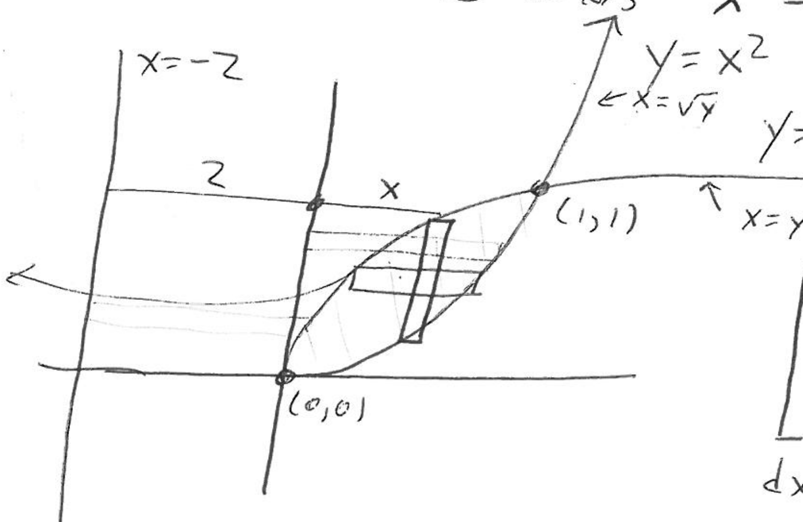
Shells

$$Vol = \int_a^b 2\pi(\text{radius})(\text{height})\,dx$$

Washers

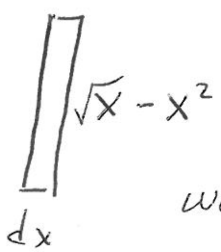
$$\int_a^b \pi(\text{out radius})^2 - \pi(\text{inner radius})^2\,d$$

Ex | Find the expression for the volume of rotating the region between $y=x^2$ and $y=\sqrt{x}$ around the axis $x=-2$.

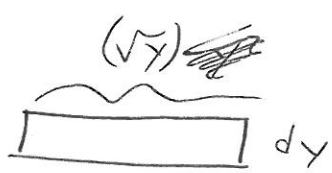


Shells

$$\int_0^1 2\pi(x+2)(\sqrt{x}-x^2)\,dx$$



washers



$$\boxed{\int_0^1 \pi(\sqrt{y}+2)^2 - \pi(y^2+2)^2\,dy}$$

Series:

Def: $\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} S_n$ where

S_n is the sequence of partial sums.

- Telescoping Series $\sum_{i=1}^{\infty} \frac{1}{i} + \frac{-1}{i+1}$

$$\sum_{i=1}^{\infty} \ln(i) - \ln(i+1)$$

- geometric series $\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i$

Ex) Find $\sum_{i=1}^{\infty} \frac{1}{i} + \frac{-1}{i+1}$ $S_1 = \frac{1}{1} - \frac{1}{2}$ $S_2 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)$

$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots \left(+\frac{1}{n} - \frac{1}{n+1}\right)$

$S_n = 1 + \frac{1}{n+1}$

$\sum \frac{1}{i} + \frac{-1}{i+1} = \lim_{n \to \infty} 1 + \frac{1}{n+1} = \boxed{1}$

Tests for Convergence and divergence.

— Test for divergence

use on $\sum \frac{1}{n^{1/n}}$, $\sum \ln(n)$

$$\sum \frac{n^2+1}{4n^2+n}$$

Ex) $\lim_{n \to \infty} \frac{n^2+1}{4n^2+n} = \frac{1}{4} \neq 0$

Hence, the the test for divergence

$$\sum_{n=1}^{\infty} \frac{n^2+1}{4n^2+n} \text{ diverges}$$

Comparison test

If $0 \le a_n \le b_n$ then
if $\sum a_n$ diverges so does $\sum b_n$
if $\sum b_n$ converges so does $\sum a_n$

Use on

$$\sum \frac{n}{n^3+1}, \quad \sum \frac{\sin^2(n)}{e^n}, \quad \sum \frac{\tan^{-1}(n)}{n^2}, \quad \sum \frac{n!}{n^n}$$

Don't use on $\sum \frac{n+2}{n^3+1}$

Ex| Show $\sum \frac{\sin^2(n)}{e^n}$ is convergent

$$0 \le \sin^2(n) \le 1$$

$$0 \le \frac{\sin^2(n)}{e^n} \le \frac{1}{e^n}$$

Look at $\sum \frac{1}{e^n}$ converges by the geometric series test

Therefor $\sum \frac{\sin^2(n)}{e^n}$ converges by comparison test.

Limit Comparison test

If $a_n > 0$ and $b_n > 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = L > 0$ then

either both $\sum a_n$ and $\sum b_n$ diverge or
both $\sum a_n$ and $\sum b_n$ converge.

Use on

$$\sum \frac{n+2}{n^3+1}, \quad \sum \frac{\ln(n)}{e^n + \ln(n)}, \quad \sum \frac{2^n + n^2}{3^n}$$

Ex| $$\lim_{n\to\infty} \frac{\frac{2^n+n^2}{3^n}}{\left(\frac{2}{3}\right)^n} = \lim_{n\to\infty} \frac{1+\frac{n^2}{2^n}}{1} = 1 + \lim_{n\to\infty} \frac{n^2}{2^n} = 1$$

By the limit comparison test, since $\sum \left(\frac{2}{3}\right)^n$ converges

and $\lim_{n\to\infty} \dfrac{\frac{2^n+n^2}{3^n}}{\frac{2^n}{3^n}} = 1$ then $\sum_{n=1}^{\infty} \frac{2^n+n^2}{3^n}$ converges.