Math 600: Integration on Chains and Stoke’s Theorem

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Outline

1. Integration on Chains
   - In Euclidean Space
   - Stoke’s Theorem in Euclidean Space
   - Green’s Theorem
   - Divergence Theorem
The subset $[0, 1]^k \subset \mathbb{R}^k$ is the **standard unit cube** in $\mathbb{R}^k$.

Let $U$ be an open subset of $\mathbb{R}^n$. A **singular $k$-cube** in $U$ is a continuous map $c : [0, 1]^k \to U$.

A singular 0-cube in $U$ is, in effect, just a point of $U$, and a singular 1-cube in $U$ is a parametrized curve in $U$. 
The standard (singular) $k$-cube $I^k : [0, 1]^k \rightarrow \mathbb{R}^k$ is the inclusion map of the standard unit cube.

A (singular) $k$-chain in $U$ is a formal finite sum of singular $k$-cubes in $U$ with integer coefficients, such as

$$2c_1 + 3c_2 - 4c_3.$$

It is clear how $k$-chains in $U$ can be added and multiplied by integers.
For each singular $k$-chain $c$ in $U$ we will define a singular $k-1$ chain in $U$ called the **boundary** of $c$ and denoted by $\partial(c)$.

We begin by defining the boundary of the standard $k$-cube $I^k : [0,1]^k \rightarrow \mathbb{R}^k$.

For each $i$ with $1 \leq i \leq k$ we define two singular $k-1$ cubes, $I^k_{(i,0)} : [0,1]^{k-1} \rightarrow [0,1]^k \subset \mathbb{R}^k$ $I^k_{(i,1)} : [0,1]^{k-1} \rightarrow [0,1]^k \subset \mathbb{R}^k$, as follows.

$$I^k_{(i,0)}(x^1, \ldots, x^{k-1}) = (x^1, \ldots, x^{i-1}, 0, x^i, \ldots, x^{k-1})$$

$$I^k_{(i,1)}(x^1, \ldots, x^{k-1}) = (x^1, \ldots, x^{i-1}, 1, x^i, \ldots, x^{k-1})$$
We call \( I^{k}_{(i,0)} \) the \((i,0)\)-face of \( I^k \) and \( I^{k}_{(i,1)} \) the \((i,1)\)-face. Of \( I^k \). Then we define

\[
\partial(I^k) = \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} I^{k}_{(i,\alpha)}.
\]

If \( c : [0,1]^k \to U \) is a singular \( k \)-cube in \( U \), we define its \((i,\alpha)\)-face by 
\( c(i,\alpha) = c \circ I^{k}_{(i,\alpha)} \), and then define

\[
\partial(c) = \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} c(i,\alpha).
\]

We extend the definition of boundary to \( k \)-chains by linearity:

\[
\partial(\sum a_i c_i) = \sum a_i \partial(c_i).
\]
**Fact:** If $c$ is a $k$-chain in $U$, show that $\partial(\partial c) = 0$. Briefly, $\partial^2 = 0$. 
Now suppose that $U$ is an open set in $\mathbb{R}^n$, that $c$ is a $k$-chain in $U$, and that $\omega$ is a differential $k$-form on $U$. We want to define the integral $\int_c \omega$ of $\omega$ over $c$, and do this in several steps.

First suppose that $\omega$ is a differential $k$-form on the unit $k$-cube $[0, 1]^k$ in $\mathbb{R}^k$. Then

$$\omega = f(x^1, \ldots, x^k)dx^1 \wedge \ldots \wedge dx^k.$$ 

In that case we define

$$\int_{[0,1]^k} \omega = \int_{[0,1]^k} f = \int_{[0,1]^k} f(x^1, \ldots, x^k)dx^1 \ldots dx^k.$$
If $\omega$ is a differential $k$-form on the open set $U$ in $\mathbb{R}^n$ and $c : [0, 1]^k \to U$ is a singular $k$-cube in $U$, we define

$$\int_c \omega = \int_{[0,1]^k} c^* \omega.$$

In other words, integration of a $k$-form over a singular $k$-cube is defined by pulling the $k$-form back to the unit $k$-cube in $\mathbb{R}^k$ and then doing ordinary integration.

In the special case that $k = 0$, a 0-form $\omega$ on $U$ is a real-valued function on $U$, and a singular 0-cube is a map $c : \{0\} \to U$ of a point into $U$. So we define

$$\int_c \omega = \omega(c(0)).$$
Finally, the integral of a \( k \)-form \( \omega \) on \( U \) over a singular \( k \)-chain \( c = \sum a_i c_i \) is defined by
\[
\int_c \omega = \sum a_i \int_{c_i} \omega.
\]

**Theorem**

**Stokes’ Theorem.** Let \( U \) be an open set in \( \mathbb{R}^n \), \( \omega \) a differential \( k - 1 \) form on \( U \), and \( c \) a singular \( k \)-chain on \( U \). Then
\[
\int_c d\omega = \int_{\partial c} \omega.
\]
**Theorem**

**Green’s Theorem.** Let $U$ be a compact region in $\mathbb{R}^2$ bounded by finitely many smooth, simple closed curves.

Let $u(x, y)$ and $v(x, y)$ be smooth functions on $U$.

Then

$$
\int_{\partial(U)} u(x, y) \ dx \ + \ v(x, y) \ dy = \int_U \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \ dx \ dy.
$$

**Proof.** Let $c$ be a singular 2-chain which covers the region $U$, so that $\partial(c)$ covers $\partial(U)$. There is some subtlety in proving the existence of $c$, but we will deal with this at a later time.
Let $\omega = u(x, y) \, dx + v(x, y) \, dy$. Then

$$d\omega = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \, dx \wedge dy.$$ 

So Green’s Theorem states that

$$\int_{\partial c} \omega = \int_c d\omega,$$

which is just a special case of Stokes’ Theorem.
**Theorem**

**Divergence Theorem.** Let $U$ be a compact region in $\mathbb{R}^3$ bounded by finitely many smooth surfaces. Let $n$ be the outward pointing unit normal vector field along $\partial(U)$. Let $V$ be a differentiable vector field on $U$. Then

$$\int_U \nabla \cdot V \ d(vol) = \int_{\partial(U)} V \cdot n \ d(area).$$

**Proof.** In words, the integral of the divergence of $V$ over the region $U$ equals the flux of $V$ through its boundary. Let

$$V = u(x, y, z)i + v(x, y, z)j + w(x, y, z)k$$

and

$$n = n_x(x, y, z)i + n_y(x, y, z)j + n_z(x, y, z)k.$$
Then \( \int_U \nabla \cdot V d(\text{vol}) = \int_U \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \, dx dy dz \)

\[ \int_{\partial(U)} V \cdot \mathbf{n} d(\text{area}) = \int_{\partial(U)} (un_x + vn_y + wn_z) \, d(\text{area}). \]

Now define a 2-form \( \omega \) on \( U \) by

\[ \omega = u(x, y, z) dy \wedge dz + v(x, y, z) dz \wedge dx + w(x, y, z) dx \wedge dy. \]

Then \( d\omega = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \, dx \wedge dy \wedge dz. \)
Thus

$$\int_U \nabla \cdot V d(vol) = \int_c d\omega,$$

where $c$ is a singular 3-chain that covers the region $U$ so that $\partial c$ covers $\partial U$, as in Green's Theorem.

Fact:

$$(un_x + vn_y + wn_z) d(area) = u dy \wedge dz + v dz \wedge dx + w dx \wedge dy.$$
Using the result of the above problem, we have that

\[ \int_{\partial U} \mathbf{V} \cdot \mathbf{n} \, d(area) = \int_{\partial U} (u n_x + v n_y + w n_z) \, d(area) \]

\[ = \int_{\partial c} (u \, dy \wedge dz + v \, dz \wedge dx + w \, dx \wedge dy) \]

\[ = \int_{\partial c} \omega. \]

So the Divergence Theorem,

\[ \int_{U} \nabla \cdot \mathbf{V} \, d(vol) = \int_{\partial U} \mathbf{V} \cdot \mathbf{n} \, d(area), \]

is a special case of Stokes’ Theorem,

\[ \int_{c} d\omega = \int_{\partial c} \omega. \]
**Theorem**

**Classical Stokes’ Theorem.** Let $S$ be a compact, smooth oriented surface in $\mathbb{R}^3$ with finitely many smooth boundary curves.

Let $\mathbf{n}$ be the unit "outward" normal vector field along $S$, and $T$ the unit tangent vector field along $\partial S$.

Let $\mathbf{V}$ be a smooth vector field defined on an open set in $\mathbb{R}^3$ which contains $S$.

Then

$$\int_S (\nabla \times \mathbf{V}) \cdot \mathbf{n} \, d(area) = \int_{\partial S} \mathbf{V} \cdot T \, d(length).$$