Math 600 Day 14: Homotopy Invariance of de Rham Cohomology

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**Differential forms on manifolds.** Let $M^n$ be a smooth manifold. A **differential p-form** $\omega$ on $M$ is a choice of a p-form $\omega(x) \in \Lambda^p T_x M$ for each $x \in M$.

If $(f, U)$ is a coordinate system on an open subset of $M$, then there is a unique differential p-form $\omega_U$ on $U$ such that $f^*(\omega(f(u))) = \omega_U(u)$ for each point $u \in U$.

If the differential p-forms $\omega_U$ are differentiable for a family of coordinate systems which cover $M$, then the differential p-form $\omega$ on $M$ is said to be differentiable (or smooth), typically of class $C^\infty$.

This definition does not depend on the choice of coordinate systems covering $M$. 
Given a smooth differential $p$-form $\omega$ on the smooth manifold $M^n$, there is a unique smooth differential $(p + 1)$-form $d\omega$ on $M$ such that for every coordinate system $(f, U)$ we have

$$f^*(d\omega) = d(f^*\omega).$$

Let $\Omega_p(M)$ denote the vector space of smooth $p$-forms on the smooth $k$-manifold $M$, and

$$\Omega^*(M) = \Omega^0(M) \oplus \Omega^1(M) \oplus \ldots \oplus \Omega^k(M)$$

the **differential graded algebra** of smooth forms on $M$. 

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Ryan Blair (U Penn)  
Math 600 Day 14: Homotopy Invariance of de Rham Cohomology  
Thursday October 28, 2010 3 / 9
Let $M^m$ be a smooth $m$-manifold, with or without boundary. Let $\Omega^k(M)$ denote the vector space of smooth $k$-forms on $M$, for $0 \leq k \leq m$. These vector spaces are connected by exterior differentiation:

$$\Omega^0(M) - d \rightarrow \Omega^1(M) - d \rightarrow \ldots - d \rightarrow \Omega^m(M).$$
Since $d^2 = 0$, the image of $d : \Omega^{k-1}(M) \to \Omega^k(M)$ is a subspace of the kernel of $d : \Omega^k(M) \to \Omega^{k+1}(M)$.

The corresponding quotient space,

$$H^k_{deR}(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\text{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))}$$

is the $k^{th}$ de Rham cohomology group of $M$ (actually a real vector space). Thus $H^k_{deR}(M)$ measures the extent to which closed $k$-forms on $M$ can fail to be exact.

The above repeats for a smooth manifold $M$ what we have already said for an open subset $U$ of Euclidean space.
Putting the vector spaces $\Omega^k(M)$ together into one package, we get

$$\Omega^*(M) = \Omega^0(M) \oplus \Omega^1(M) \oplus ... \oplus \Omega^k(M),$$

the differential graded algebra of smooth forms on $M$. The multiplication comes from the exterior product

$$\wedge : \Omega^p(M) \times \Omega^q(M) \rightarrow \Omega^{p+q}(M).$$
Note that

1. \( d\phi_1 = 0 \) and \( d\phi_2 = 0 \) implies \( d(\phi_1 \wedge \phi_2) = 0 \).
   That is, \( \text{closed} \wedge \text{closed} = \text{closed} \).

2. \( \phi_1 = d\mu_1 \) and \( d\phi_2 = 0 \) implies \( \phi_1 \wedge \phi_2 = d(\mu_1 \wedge \phi_2) \).
   That is, \( \text{exact} \wedge \text{closed} = \text{exact} \).

Hence the exterior product at the level of differential forms induces a cup product at the level of de Rham cohomology:

\[ \cup : H^p_{\text{deR}}(M) \times H^q_{\text{deR}}(M) \rightarrow H^{p+q}_{\text{deR}}(M) \]

This cup product makes

\[ H^*_{\text{deR}}(M) = H^0_{\text{deR}}(M) \oplus H^1_{\text{deR}}(M) \oplus \ldots \oplus H^m_{\text{deR}}(M) \]

into a graded algebra, called the de Rham cohomology algebra of \( M \).
Induced mappings.

A smooth map $f : M \to N$ between smooth manifolds induces maps in the other direction, $f^* : \Omega^k(N) \to \Omega^k(M)$ between smooth $k$-forms. These induced maps commute with exterior products,

$$f^*(\phi_1 \wedge \phi_2) = f^*(\phi_1) \wedge f^*(\phi_2),$$

and hence assemble to a homomorphism of graded algebras

$$f^* : \Omega^*(N) \to \Omega^*(M).$$
If $\phi$ is a closed k-form on $N$, then $f^*(\phi)$ is a closed k-form on $M$ because $d(f^*\phi) = f^*(d\phi) = f^*(0) = 0$.

If $\phi$ is an exact k-form on $N$, say $\phi = d\mu$, then $f^*(\phi)$ is an exact k-form on $M$ because $d(f^*\mu) = f^*(d\mu) = f^*(\phi)$.

Hence we get an induced map $f^*: H^k_{deR}(N) \to H^k_{deR}(M)$, defined by $f^*([\phi]) = [f^*\phi]$, where $[\phi]$ represents the cohomology class of the closed k-form $\phi$. These induced linear maps assemble to an algebra homomorphism

$$f^*: H^*_{deR}(N) \to H^*_{deR}(M)$$

between the graded de Rham cohomology algebras.