Math 600 Day 11: Multilinear Algebra

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Outline

1 Multilinear Algebra
$V = \text{ vector space (typically finite dim’l) over } \mathbb{R}$

$V^k = \text{k-fold product } V \times \ldots \times V$

A function $T : V^k \to \mathbb{R}$ is said to be multilinear if it is linear in each variable when the other $k - 1$ variables are held fixed. Such a multilinear function $T$ is called a $k$-tensor on $V$. 
Example. An inner product on $V$ is a 2-tensor which is symmetric and positive definite.

$\mathcal{T}^k(V) = \text{set of all } k\text{-tensors on } V, \text{ is a vector space over } \mathbb{R} \text{ in the natural way.} $

Note that $\mathcal{T}^1(V)$ is just the dual space $V^*$.  

If $S \in \mathcal{T}^k(V)$ and $T \in \mathcal{T}^r(V)$, we define a tensor product $S \otimes T \in \mathcal{T}^{k+r}(V)$ by

$$(S \otimes T)(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+r}) = S(v_1, \ldots, v_k)T(v_{k+1}, \ldots, v_{k+r}).$$

Note that $S \otimes T \neq T \otimes S$.

**Tensor Equalities.**

$$(S_1 + S_2) \otimes T = (S_1 \otimes T) + (S_2 \otimes T)$$

$$S \otimes (T_1 + T_2) = (S \otimes T_1) + (S \otimes T_2)$$

$$(aS) \otimes T = S \otimes (aT) = a(S \otimes T)$$

$$(S \otimes T) \otimes U = S \otimes (T \otimes U).$$
Exercise. Let $v_1, \ldots, v_n$ be a basis for $V$, and let $\varphi_1, \ldots, \varphi_n$ be the dual basis for $V^* = T^1(V)$, meaning that $\varphi_i(v_j) = \delta_{ij}$. Show that the set of all $k$-fold tensor products

$$\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}, 1 \leq i_1, \ldots, i_k \leq n$$

is a basis for $T^k(V)$, which therefore has dimension $n^k$. 
Definition

If $f : V \to W$ is a linear map, then a linear map $f^* : T^k(W) \to T^k(V)$ is defined by

$$(f^* T)(v_1, \ldots, v_k) = T(f(v_1), \ldots, f(v_k)).$$

When $k = 1$, this is just the familiar transpose or adjoint of a linear map.

Note that $f^*(S \otimes T) = f^* S \otimes f^* T$. 
Definition

A $k$-tensor $\omega \in \mathcal{T}^k(V)$ is **alternating** if

$$\omega(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\omega(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

for all $v_1, \ldots, v_k \in V$. That is, $\omega$ changes sign when exactly two of its variables are interchanged.

An alternating $k$-tensor is called a **$k$-form**.

The set of $k$-forms is denoted by $\Lambda^k(V)$, and is a subspace of $\mathcal{T}^k(V)$. 

Any k-tensor can be turned into a k-form:

\[ \text{Alt}(T)(v_1, \ldots, v_k) = \text{defn} \left( \frac{1}{k!} \right) \sum_{\sigma \in S_k} (-1)^\sigma T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) , \]

where \( S_k \) is the symmetric group of all permutations of the numbers 1 to \( k \) and \((-1)^\sigma\) is the sign of the permutation \( \sigma \).

**Exercise.** Show that \( \text{Alt}(T) \) really is alternating.

**Other Facts**
(a) If \( T \) is already alternating, \( \text{Alt}(T) = T \).
(b) \( \text{Alt}(\text{Alt}(T)) = \text{Alt}(T) \).
Definition

If \( \omega \in \Lambda^k(V) \) and \( \eta \in \Lambda^r(V) \), the **wedge product** \( \omega \wedge \eta \in \Lambda^{k+r}(V) \) is defined by

\[
\omega \wedge \eta = \frac{(k+r)!}{k!r!} \text{Alt}(\omega \otimes \eta).
\]

For example, if \( \varphi_1 \) and \( \varphi_2 \) are 1-forms, we have

\[
(\varphi_1 \wedge \varphi_2)(v_1, v_2) = \varphi_1(v_1)\varphi_2(v_2) - \varphi_1(v_2)\varphi_2(v_1).
\]

Note that for a 1-form \( \varphi \) we have \( \varphi \wedge \varphi = 0 \).
**Exercise.** Let $\omega$ be a $k$-form and $\eta$ an $r$-form. Show that

\[
(\omega \wedge \eta)(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+r}) = \sum_{\sigma \in S'} (-1)^\sigma \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \eta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+r)}),
\]

where $S'$ is the subset of the symmetric group $S_{k+r}$ consisting of all permutations $\sigma$ such that

\[
\sigma(1) < \ldots < \sigma(k) \text{ and } \sigma(k+1) < \ldots < \sigma(k+r).
\]
Properties of the wedge product:

1. $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$
2. $\omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2$
3. $(a\omega) \wedge \eta = \omega \wedge (a\eta) = a(\omega \wedge \eta)$
4. $\omega \wedge \eta = (-1)^{kr} \eta \wedge \omega$
5. $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$. 
Problem. Show that

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k + r + s)!}{k!r!s!} \text{Alt}(\omega \otimes \eta \otimes \theta),$$

where $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^r(V)$ and $\theta \in \Lambda^s(V)$.

This is harder.

So now we can drop the parentheses, and simply write $\omega \wedge \eta \wedge \theta$, and likewise for higher order products.
Exercise. Let $v_1, ..., v_n$ be a basis for $V$, and let $\varphi_1, ..., \varphi_n$ be the dual basis for $V^* = T^1(V)$. Show that the set of all $\varphi_{i_1} \wedge ... \wedge \varphi_{i_k}$, with $1 \leq i_1 < i_2 < ... < i_k \leq n$ is a basis for $\Lambda^k(V)$, which therefore has dimension

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Show, in fact, that $(\varphi_{i_1} \wedge ... \wedge \varphi_{i_k})(v_{i_1}, ..., v_{i_k}) = 1$.

Note in particular that $\Lambda^n(V)$ is one-dimensional.