# Homotopy Continuation Methods

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## Introduction

- homotopic if one continuous function can be continuously deformed into another.
- $t$ ranges from 0 to 1 (Decarolis, Mayer, Santamaria).

## Two Interesting Homotopies

- Fixed Point Homotopy is a continuation method that gradually deforms $(x - x_0)$ into $f(x)$. The equation used for Fixed Point Homotopy is: $H(x,t) = (1-t)(x-x_0) + tf(x)$, for some $x_0$.
- Newton's Homotopy is a continuation method that for $t = 0$ has its zero at $x = x_0$ while at $t = 1$, the homotopy function is $f(x)$ (Decarolis, Mayer, Santamaria).
  The equation used is: $H(x,t) = f(x) - (1-t)f(x_0)$, for some $x_0$ (Decarolis, Mayer, Santamaria).

## Consider

$$f(x) = 2x - 4 + sin(2\pi x)$$

- $x_0$ is the initial guess for $x$. In this problem, the initial guess is 0 (Judd 1998).
- Figures 1 and 2 show the path connecting the zero of $H(x,0)$ to that of $H(x,1)$. This zero is found at $x = 2.0008$ by the Fixed Point Homotopy method (Decarolis, Mayer, Santamaria).
- Newton's root finding computer algorithm solves this problem to give a root of exactly 2.
- Homotopic solution isn't 2, but it gives a good approximation for an initial guess in the algorithm.
- If one were to use 2.0008 as an initial guess for x, the solution will be found more rapidly, while using Newton's root finding algorithm (Decarolis, Mayer, Santamaria).

Figure 1: the Fixed Point Homotopy

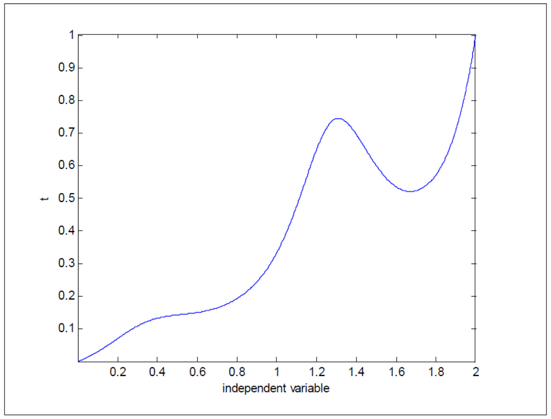


Figure 1 shows the Fixed Point Homotopy. Points are tested along the way in the algorithm, and get closer to zero. Once the homotopy equation gets close enough to zero, an answer is produced. If points are tested along the curve, they will show how close the homotopy equation gets to zero.

Figure 2: the Newton Homotopy

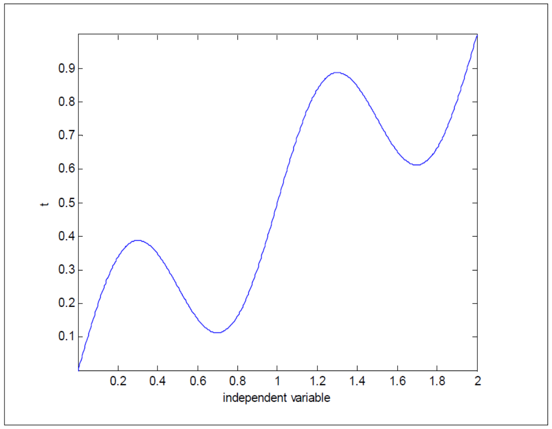


Figure 2 shows Newton's Homotopy for the equation given. For $t = 0$, Newton's Homotopy has its' zero at $x = x_0$ while at $t = 1$, the homotopy function is $f(x)$ (Decarolis, Mayer, Santamaria). The root at $t = 1$ is $x = 2$. Notice that the path is different than that of the fixed point homotopy.

## Results

- The exact answer to this problem is $x = 2$.
- When we compute $H(2,1)$ in the Fixed Point Homotopy, with initial guess $x_0 = 0$, we get: $H(2,1) = (1-1)(2-0) + 1f(2) = f(2) = 0$.
- Likewise, when we compute H(2,1) in Newton's Homotopy, with initial guess $x_0 = 0$, we get: $H(2,1) = f(2) - (1-1)f(0) = f(2) = 0$.
- Note that the only homotopy that found 2 exactly was Newton's Homotopy.

## Summary

- Newton's Homotopy and Fixed Point Homotopy are two interesting continuation methods that help to deform two continuous functions from two topological spaces into each other.
- It is important, since homotopies are numerical approximations, to use root finding computer algorithms along with the chosen homotopy.

## Conclusions and Acknowledgements

- When individual points on the figures are substituted into the homotopy equation, H, they can yield positive or negative solutions. The path that the graph follows eventually finds a zero.
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