# Newton's Method and Fractal Images

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## Introduction

•Newton's Method is an iterative formula used to find the roots of a function.
•By using Newton's method on complex functions fractal images can be generated.
•A fractal is a pattern which repeats itself on multiple scales. Fractals are common in nature.

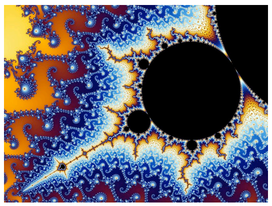

**Figure 1** Mandelbrot Set
Copyright Wolfgang Beyer
Retrieved 12/06/09 wikipedia.org

**Figure 2** Romanesco broccoli

## Methods

Given an initial starting point z in the complex plane
f(z) is a complex function
The (n+1)$^{th}$ approximation to a root of f(z) is given by

- **Standard Newton's Method**

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}$$

This method has quadratic convergence to simple roots, but only linear convergence to multiple roots.

- **Relaxed Newton's Method**

Given a root of order k, apply Newton's method to $\sqrt[k]{f(z)}$

$$z_{n+1} = z_n - k\frac{f(z_n)}{f'(z_n)}$$

This method is used to get better convergence at multiple roots, but the order must be known.

- **Multiple Root Method**

$$z_{n+1} = z_n - \frac{f(z_n)f'(z_n)}{[f'(z_n)]^2 - f''(z_n)f(z_n)}$$

Guaranteed quadratic convergence at every root, but involves the second derivative.

- Other Methods: Collatz, Schröder, and König

**To Generate Fractal Images:**

1. Selecting one of the methods, take a region in the complex plane and generate a large amount of lattice points in that region.
2. Use each lattice point as the starting point z for the selected method. Group lattice points that converge to the same root.
3. Assign each group of lattice points a different color. Colors should be distinct.
4. Plot each lattice point with its corresponding color.
5. The number of steps until the point converged can be used optionally to give shades of color.

## Results

•A **basin of attraction** is defined to be the set of all points that converge to the same root
•By coloring each basin of attraction a different color, the boundaries between the basins are easy to see.
•**Julia sets** are defined as the boundaries between the basins of attraction.
•These boundary points are points that do not converge in any of Newton's Methods, and form fractal images.
•It is of note that the boundary of any one of the basins is also the boundary of all the other basins.

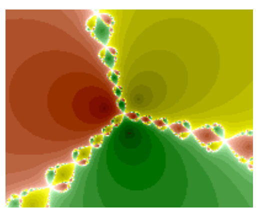

**Figure 3** Standard Newton's Method
f(z)=(z+1+2i)(z+2+2i)(z+0.634+0.634i)

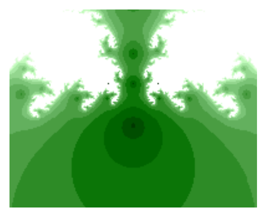

**Figure 4** Relaxed Newton's Method
f(z)=(z-1-3i)(z-3-3i)(z-2-1.268i)$^2$

Source: Figures 3 and 4 from Reference 2

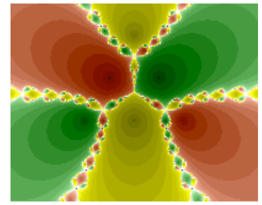

**Figure 5** Multiple Root Method
f(z)=(z-1-3i)(z-3-3i)(z-2-1.268i)

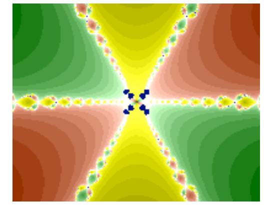

**Figure 6** Previous figure zoomed out
Notice how the pattern repeats

Figures 5 and 6 from Reference 2

## Summary

•Using Newton's method on complex functions yields fractal images.
•These images show the basins of attraction for each root of the function.
•The boundaries between basins form Julia sets which have a fractal appearance.
•Further magnification of the boundaries yields more patterns similar to the whole.

## Conclusions

•Newton's method is incredibly versatile.
•Generating fractal images from Newton's method illustrates the link between math and art.
•Fractals have important applications in many fields of study, and continued exploration will increase understanding of them.

## Acknowledgments

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## References

1. Kincaid, D., & Cheney, W. (2002). *Numerical Analysis: Mathematics of Scientific Computing* (3rd ed.). Providence, Rhode Island: American Mathematical Society.
2. Wang, X., & Wang, T. (2007). Julia Sets of Generalized Newton's Method. *Fractals* , *15* (4), 323-336.