Ways to determine GPS

Krista Katayama

Department of Mathematics and Statistics
California State University, Long Beach

May 14, 2009

Outline

What is GPS?

GPS: *Global Positioning System*

- a satellite based navigation system
- computes location using radio frequencies

Geometric model

- variant of the 3-D triangulation
  - position determined based on distance from 3 other points
- position is GPS receiver location
- 3 other points are satellites

[1]

How to find distances?

- position is restricted to lie on a sphere centered at the fixed point (satellite position)
- must be true $\forall$ fixed point (i.e. $\forall$ satellite)

[1]

Remember:

- intersection of 2 spheres is a circle
- generally intersection of 3 spheres leads to 2 points

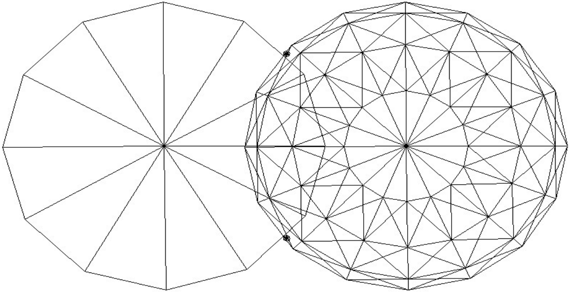


Figure: from Wikipedia article on GPS

How to choose between 2 points?

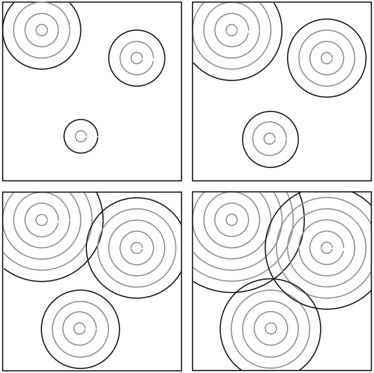
Want position to be on the surface of the earth

- one will land in space or within the earth

$\Longrightarrow$ take the position location that makes the most sense

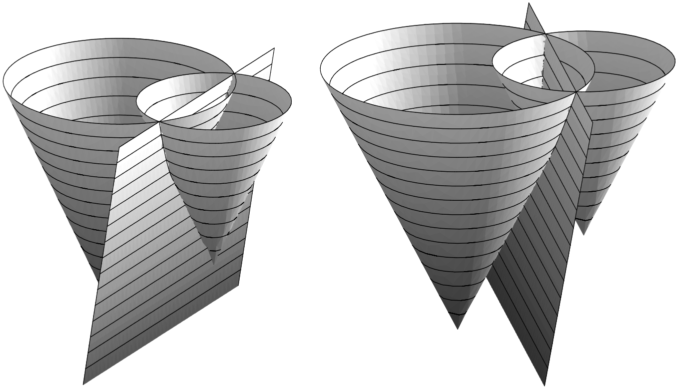
Problem:

- GPS has a triangulation in space AND time
- new visualization: 3-D space-time
  - think of horizontal plane representing space
  - think of vertical plane representing time



Use cones instead of spheres

- intersection of 2 cones lies in a plane
- having 3 cones would result in 2 planes
  - intersection of 3 cones results in a line
- need another cone to intersect line



How can we find the position?

- now need 4 satellites
- leads to 4 similar equations (based on distance)
- corresponds to solving underdetermined system of linear equations

[1]

Quick Example

Table 1. Satellite data.

Satellite	Position	Time
1	(1, 2, 0)	19.9
2	(2, 0, 2)	2.4
3	(1, 1, 1)	32.6
4	(2, 1, 0)	19.9

- time is the time sent
- Assumption
  - signals travel at the speed of light $\left(0.047 \frac{\text{earth radii}}{\text{millisecond}}\right)$

[1]

Getting the distance

- take 1st satellite, get 2 equations for distance:

$$\begin{aligned} d &= 0.047\,(t - 19.9) \\ d &= \sqrt{(x-1)^2 + (y-2)^2 + (z-0)^2} \end{aligned}$$

$$\Longrightarrow \; 0.047\,(t - 19.9) = \sqrt{(x-1)^2 + (y-2)^2 + (z-0)^2}$$

[1]

Expand and rearrange

$$2x + 4y - 2\left(0.047^2\right)(19.9)^2\, t = 1^2 + 2^2 + 0^2 + x^2 + y^2 + z^2 - 0.047^2 t^2$$

All 4 equations have a similar form

[1]

Underdetermined System

- remove quadratic terms from equations
  - subtract the 1st equation from the others

$$2x - 4y + 4z + 2\left(0.047^2\right)(17.5)\,t = 8 - 5 + 0.047^2\left(19.9^2 - 2.4^2\right)$$
$$0x - 2y + 2z - 2\left(0.047^2\right)(12.7)\,t = 3 - 5 + 0.047^2\left(19.9^2 - 32.6^2\right)$$
$$2x - 2y + 0z + 2\left(0.047^2\right)(0)\,t = 5 - 5 + 0.047^2\left(19.9^2 - 19.9^2\right)$$

- notice there are 4 variables and 3 equations
  - will not get a unique solution
- can get 3 variables in terms of the 4th

[1]

Matrix form

The system has the form:

$$\begin{bmatrix} 2 & -4 & 4 & .077 & 3.86 \\ 0 & -2 & 2 & -.056 & -3.47 \\ 2 & -2 & 0 & 0 & 0 \end{bmatrix}$$

Its reduced row echelon form is:

$$\begin{bmatrix} 1 & 0 & 0 & .095 & 5.41 \\ 0 & 1 & 0 & .095 & 5.41 \\ 0 & 0 & 1 & .067 & 3.67 \end{bmatrix}$$

[1]

Solution:

The general solution is:

$$x = 5.41 - .095t, \quad y = 5.41 - .095t, z \quad = 3.67 - .067t, \quad t \text{ free}$$

After substituting into the original equation, get

$$0.02t^2 - 1.88t + 43.56 = 0$$

- get 2 solutions: $t = 43.1, 50.0$
- $\Rightarrow$ $(1.317, 1.317, 0.790)$, $(.667, .667, .332)$

Remember units in earth radii

- 1^{st} solution puts position outside the earth
- 2^{nd} solution is the position

[1]

Remarks:

- this method not actually used to determine position with GPS
- does not take into account errors

Things to consider:

Factor in errors

- Earth's atmosphere is not a vacuum
- cannot use the speed of light for velocity

Satellite's position

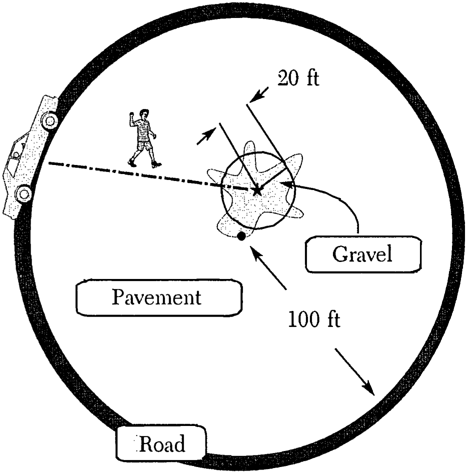
- consider angle of signals

Will see another error appear: clock errors

[2]

Look at 2-D model

- A different velocity is needed for area around position



Initial Setup

A person stands on a gravel plot within a circular lot

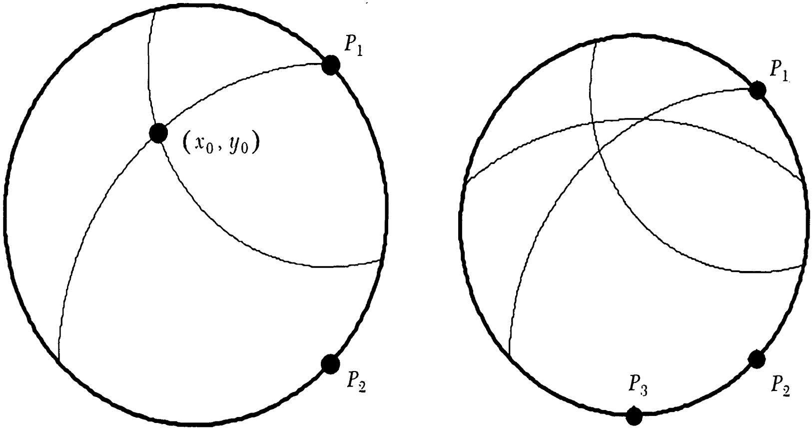
- lot has a radius of 100 ft
- mean distance from person's position to edge of gravel is 20 ft
- cars drive on a road that borders the circular lot

Messengers leave from the cars on the road and walk to the person

- move at 5 ft/sec on the pavement
- move at 4 ft/sec on gravel

[2]

Messenger paths



[2]

- Δt: time difference between departure and arrival
- ε: fixed error of watch (in seconds)

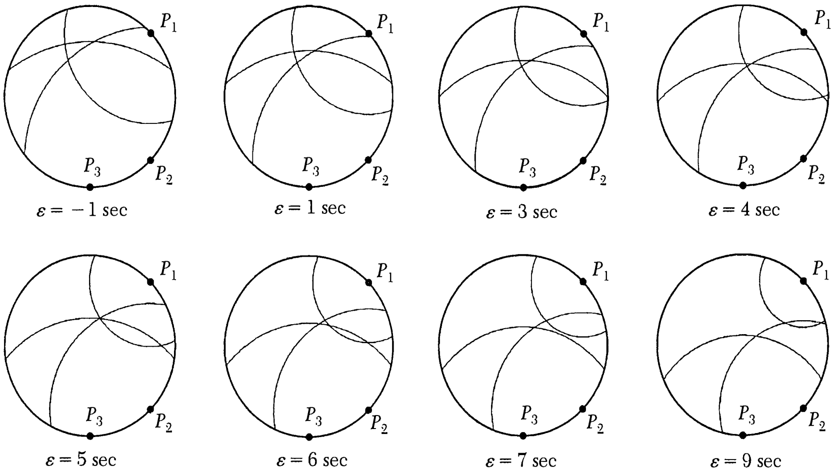
Estimate of the distance travelled:

$$d(\Delta t, \varepsilon) = 20 \text{ ft } + (\Delta t \text{ sec } - \varepsilon \text{ sec } - 5 \text{ sec})\, 5 \frac{\text{ft}}{\text{sec}}$$

$$\left\{ \begin{array}{rcl} (x_0 - 70.7)^2 + (y_0 - 70.7)^2 & = & d(20.2, \varepsilon)^2 \\ (x_0 - 70.7)^2 + (y_0 + 70.7)^2 & = & d(29.5, \varepsilon)^2 \\ (x_0 - 0)^2 + (y_0 + 100)^2 & = & d(32.2, \varepsilon)^2 \end{array} \right\}$$

[2]

Clock Errors



Solve numerically for the position starting with $\varepsilon = 0$

- get a solution when position is inside the lot

Change the parameters

- circular lot $\Longrightarrow$ region within satellite orbits
- cars $\Longrightarrow$ satellites
- messengers $\Longrightarrow$ radio waves
- gravel $\Longrightarrow$ Earth's atmosphere
- origin $\Longrightarrow$ center of the Earth

Will need at least 4 satellites

[2]

Change the variables

- $S_i = (X_i, Y_i, Z_i) \implies$ position of satellite
- $T_i \implies$ time when satellite i transmits a signal
- $T_i' \implies$ time signal is received
- $\Delta t_i \implies$ travel time
- $\varepsilon \implies$ clock time error of the receiver

Typically only one value of ε will allow the spheres to have a common point

[2]

New system to solve

$$\left\{\begin{array}{rcl}(x_0 - X_1)^2 + (y_0 - Y_1)^2 + (z_0 - Z_1)^2 & = & d(\Delta t_1, \varepsilon)^2 \\ (x_0 - X_2)^2 + (y_0 - Y_2)^2 + (z_0 - Z_2)^2 & = & d(\Delta t_2, \varepsilon)^2 \\ (x_0 - X_3)^2 + (y_0 - Y_3)^2 + (z_0 - Z_3)^2 & = & d(\Delta t_3, \varepsilon)^2 \\ (x_0 - X_4)^2 + (y_0 - Y_4)^2 + (z_0 - Z_4)^2 & = & d(\Delta t_4, \varepsilon)^2\end{array}\right\}$$

[2]

What to do with answer?

Need to change answer (in rectangular coordinates) to spherical coordinates

Will then have:

- latitude
- longitude
- altitude (above sea level)

What is expected of the receiver?

Expectations:

- receive satellite time and position information
- maintain a steady clock (not necessarily accurate)
- find 4 satellites with “good” position ranges
- approximate a numerical solution for 4 equation system
- transform coordinates

Note: with today’s technology, these expectations are not unreasonable.

[2]

Variability of Positions

Position estimation varies with repeated attempts

Caused by:

- random measurement errors
- selection of different satellites
- atmosphere effects

[2]

Ways to deal with errors

PPS - Precise Positioning Service

- uses multiple signals
- for military use only

DGPS - Differential GPS

- 2 receivers
- 1 has known fixed position
- 1 moves around

[2]

Design of GPS

Original Design:

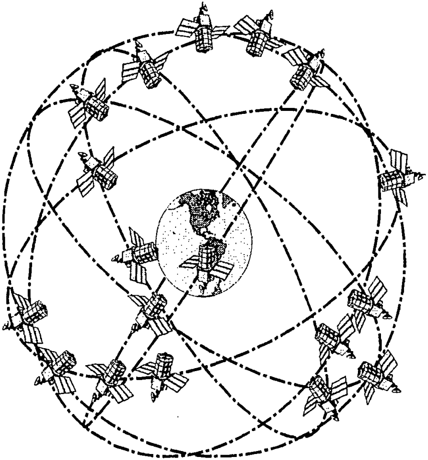
- 18 satellites
- 6 orbits
- $\implies$ 3 satellites per orbit

Current Design (as of 1998):

- 4 satellites in each orbit
- same general setup as the original

[2]

Design of GPS



[2]

Different Carriers and Codes:

Carriers - there are two carrier radio waves:

- L1, with frequency 1575.42 MHz
- L2, with frequency 1227.6 MHz

Pseudo-random Codes that are superimposed on the carriers:

- On the L1 carrier:
  - C/A code: Coarse Acquisition code
  - P-code: Precision code
- On the L2 carrier:
  - P-code: Precision code

Note: the C/A code is for civilian users; only authorized users have access to the P-code

[3]

What are the carriers and codes used for?

What information is given from the carrier?

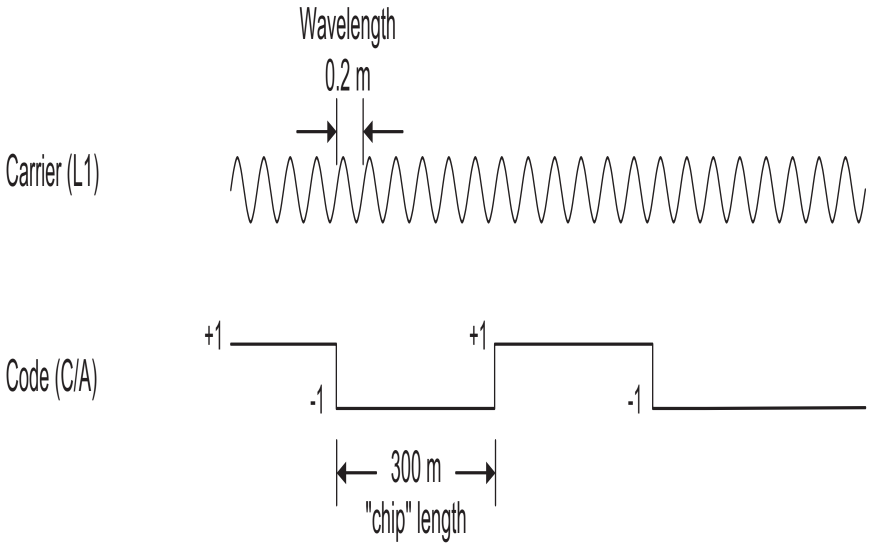
- position of the satellite
- the exact time the signal was transmitted

How is the code used?

- Allows a GPS receiver to measure the travel time of the signal from the satellite to the receiver.

[3]

What do they look like?



How to read signals from different satellites on the same frequency?

Each satellite is given its own unique pseudo-random code!

- avoids jamming with other signals
- avoids receiver comparison to wrong signal

How to determine position?

Follow these general steps:

1. Extract information from the satellite signal
2. Compare information with receiver information
3. Determine Δt from information correlation
4. Compute the distance from the satellite to the receiver
5. Repeat for every “good” satellite

How is the information used?

There are 2 types of methods used to determine position:

- Code Pseudorange
- Carrier Phase

[3]

Which method is better?

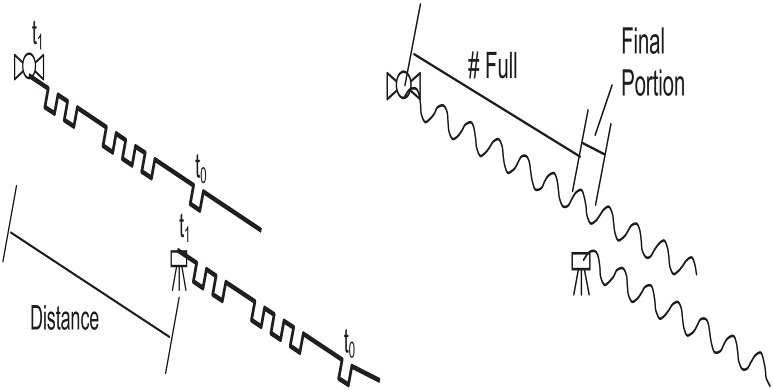
Code Pseudorange gives an approximation to the true range between the receiver and satellite using the C/A pseudo-random code

- relatively easy calculation
- results are not very accurate

Carrier Phase gives an approximation to the true range between the receiver and satellite using one of the carrier frequencies

- uses the L1 carrier for non-military receivers
- requires a series of observations
- can get better accuracy

Comparing signals



- receiver generates signal at same time as satellite
- carrier frequency hard to count since it's so uniform
- cycles of code are wide - plenty of room to 'slop'

Best Results

Use both

- Use codepseudorange to get "close"
- Use carrier signal to get "good" accuracy

Components of DGPS:

1. Space Segment
   - satellites which broadcast the signal
2. Control Segment
   - steers the whole system
3. User Segment
   - many types of receivers

[3]

General Idea

- Use 2 receivers
  - 1 stationary
  - 1 moving
- stationary position is known exactly
- other is estimated
- use stationary receiver to send out “correction” term to other receiver

[3]

Expected errors:

- ionospheric range error
- tropospheric range error
- satellite clock range error
- receiver clock range error
- multipath error
- noise

[3]

Assumptions

- distance ("baseline") between 2 receivers is short, i.e. $\approx$ 30km

Why?

- receivers then have relatively the same ionospheric and tropospheric refraction errors
- these errors are essentially eliminated when taking the difference of the 2 signals
- also gets rid of satellite clock error

This is considered "Single Differencing"

[3]

Double Differencing

- Find single differenced measurements from all “good” satellites
- choose one satellite to be the “reference” satellite
- take the difference of each single differenced measurement from the “reference” satellite

[3]

Pros/Cons of Double Differencing

Pro:

- eliminates the 2 receivers' clock errors

Cons:

- numerically slightly dubious (makes measurements correlated)
- gives unnecessary prominence to "reference" satellite

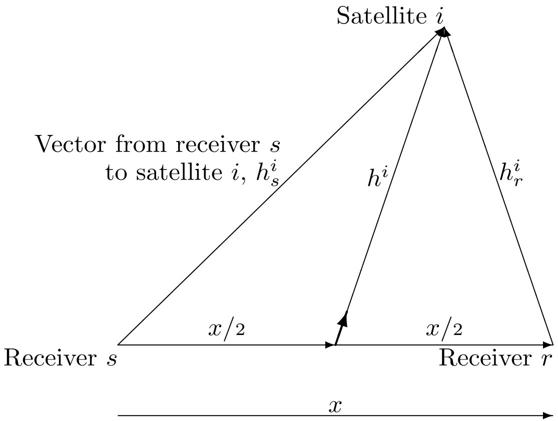
[3]

Parameters

- use carrier phase measurement (L1 signal)
- use single differencing (with assumptions)
- use recursive least squares approach to estimate position
- assume number of "good" satellites remain constant

[3]

Breaking down the math model



- Goal: find baseline vector **x**

[3]

Breaking down the math model

- h_s^i is the vector from receiver s to satellite i
- e^i is the unit vector from the midpoint of the baseline to satellite i
- ρ_s^i is the range in wavelengths from receiver s to satellite i
- λ is the wavelength
- $\mu^i = \frac{\|h_s^i + h_r^i\|}{\|h_s^i\| + \|h_r^i\|} \approx \frac{1}{1 + .28 \times 10^{-6}}$ (normally rounded to 1)

[3]

Finding $\rho_s^i,\ \rho_r^i$

Get this equation:

$$\left(\mu^i e^i\right)^T x = \lambda\left(\rho_s^i - \rho_r^i\right)$$

- initially find fractional phase difference (part of wavelength) between generated and received signal
- track how phase difference changes

[3]

New variables

$\eta_s^i(t_k)$: carrier phase measurement

- from receiver s to satellite i at time t_k

α_s^i: "integer ambiguity"

- initial number of full cycles between satellite i and receiver s at t_1

Ideally want

$$\rho_s^i(t_k) = \eta_s^i(t_k) + \alpha_s^i$$

but need to factor in errors

$$\eta_s^i(t_k) + \alpha_s^i = \rho_s^i(t_k) - \iota_s^i(t_k) + \tau_s^i(t_k) + \beta^i\left(t_k - t_k^i\right) + \beta_s^i(t_k) + \nu_s^i(t_k)$$

[3]

Take the difference between stationary and moving receiver carrier phase measurements

$$\eta_k^i = \lambda^{-1} \left(\mu_k^i e_k^i\right)^T x_k - \alpha^i + \beta_k + \nu_k^i$$

Assume ν_k^i are unbiased independently distributed noises for different satellites and epochs

[3]

Simplify

$$y_k = E_k x_k - a + e\beta_k + \upsilon_k$$

where $\upsilon_k \sim \mathcal{N}\left(0, \sigma^2 I_m\right)$

Add in all epochs (time steps) up to k to get

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ y_k \end{bmatrix} = \begin{bmatrix} e & E_1 & & & & & -I_m \\ & & e & E_2 & & & -I_m \\ & & & & \cdot \quad \cdot & & \cdot \\ & & & & e & E_k & -I_m \end{bmatrix} \begin{bmatrix} \beta_1 \\ x_1 \\ \beta_2 \\ x_2 \\ \cdot \\ \cdot \\ \beta_k \\ x_k \\ a \end{bmatrix} + \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \cdot \\ \upsilon_k \end{bmatrix}$$

Remember, want to find x_ks

[3]

Using LS approach

- recall: coefficient matrix must have full column rank to get a unique LS solution
- use orthogonal transformations of single differences
- Using Householder transformations

$$P \equiv I - u\left(\frac{2}{u^T u}\right)u^T, \qquad u \equiv e_1 - e/\sqrt{m}$$

[3]

Using LS approach

- thus P has the form:

$$P = \begin{bmatrix} \frac{1}{\sqrt{m}} & \frac{e^T}{\sqrt{m}} \\ \frac{e}{\sqrt{m}} & I_{m-1} - \frac{ee^T}{m-\sqrt{m}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{m}} & \frac{1}{\sqrt{m}} & \frac{1}{\sqrt{m}} & \cdot \\ \frac{1}{\sqrt{m}} & 1 - \frac{1}{m-\sqrt{m}} & -\frac{1}{m-\sqrt{m}} & \cdot \\ \frac{1}{\sqrt{m}} & -\frac{1}{m-\sqrt{m}} & 1 - \frac{1}{m-\sqrt{m}} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$= \begin{bmatrix} p_1, & P_2 \end{bmatrix}$$

[3]

Apply P^T to:

$$\begin{bmatrix} y_1 \\ y_2 \\ . \\ y_k \end{bmatrix} = \begin{bmatrix} e & E_1 & & & & & -I_m \\ & & e & E_2 & & & -I_m \\ & & & & . \quad . & & . \\ & & & & & e \quad E_k & -I_m \end{bmatrix} \begin{bmatrix} \beta_1 \\ x_1 \\ \beta_2 \\ x_2 \\ . \\ . \\ \beta_k \\ x_k \\ a \end{bmatrix} + \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ . \\ \upsilon_k \end{bmatrix}$$

[3]

Apply P^T to:

The result is:

$$\begin{bmatrix} p_1^T y_1 \\ p_1^T y_2 \\ \vdots \\ p_1^T y_k \\ P_2^T y_1 \\ P_2^T y_2 \\ \vdots \\ P_2^T y_k \end{bmatrix} = \begin{bmatrix} 1 & & & p_1^T E_1 & & & & \\ & 1 & & & p_1^T E_2 & & & \\ & & \ddots & & & \ddots & & \\ & & & 1 & & & p_1^T E_k & \\ & & & P_2^T E_1 & & & & I_{m-1} \\ & & & & P_2^T E_2 & & & I_{m-1} \\ & & & & & \ddots & & \vdots \\ & & & & & & P_2^T E_k & I_{m-1} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_k \\ x_1 \\ x_2 \\ \vdots \\ x_k \\ d \end{bmatrix} + \begin{bmatrix} p_1^T \upsilon_1 \\ p_1^T \upsilon_2 \\ \vdots \\ p_1^T \upsilon_k \\ P_2^T \upsilon_1 \\ P_2^T \upsilon_2 \\ \vdots \\ P_2^T \upsilon_k \end{bmatrix}$$

[3]

Simpler model

Want to find x_ks, so solve

$$\begin{bmatrix} P_2^T y_1 \\ P_2^T y_2 \\ \cdot \\ P_2^T y_k \end{bmatrix} = \begin{bmatrix} P_2^T E_1 & & & I_{m-1} \\ & P_2^T E_2 & & I_{m-1} \\ & & \cdot & \cdot \\ & & P_2^T E_k & I_{m-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_k \\ d \end{bmatrix} + \begin{bmatrix} P_2^T \upsilon_1 \\ P_2^T \upsilon_2 \\ \cdot \\ P_2^T \upsilon_k \end{bmatrix}$$

where $d = -P_2^T a$

[3]

Simpler model

Note: Coefficient matrix has size $k\,(m-1) \times (3k+m-1)$

$\Longrightarrow$ has full column rank if $k\,(m-1) \geq 3k+m-1$, i.e.

$$m \geq 4 + \frac{3}{k-1}$$

Satellite requirements.

# of epochs (k)	2	3	4	5	≥ 6
Minimum # of satellites	7	6	5	5	5

[3]

Problem

When switching from a to d, lose integer nature

- want to keep, so rewrite system
- define double difference integer ambiguity as a^{DD}
- rewrite P_2^T and d such that:

$$P_2^T \equiv -FJ, \qquad d \equiv -P_2^T a = FJa = Fa^{DD}$$

where

$$F \equiv I_{m-1} - \frac{ee^T}{m - \sqrt{m}}, \qquad J \equiv [e, -I_{m-1}]$$

[3]

Remedy

replace d to get:

$$\begin{bmatrix} P_2^T y_1 \\ P_2^T y_2 \\ . \\ P_2^T y_k \end{bmatrix} = \begin{bmatrix} P_2^T E_1 & & & & F \\ & P_2^T E_2 & & & F \\ & & . & & . \\ & & & P_2^T E_k & F \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_k \\ a^{DD} \end{bmatrix} + \begin{bmatrix} P_2^T \upsilon_1 \\ P_2^T \upsilon_2 \\ . \\ P_2^T \upsilon_k \end{bmatrix}$$

[3]

QR Factorization

Using $Q_j^T = \begin{bmatrix} U_j \\ W_j \end{bmatrix}$,

$$Q_j^T\left(P_2^T E_j\right) = \begin{bmatrix} R_j \\ 0 \end{bmatrix}, \qquad Q_j^T\left(P_2^T y_j\right) = \begin{bmatrix} u_j \\ w_j \end{bmatrix}$$

Applying this to the previous system results with:

$$\begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ u_k \\ w_1 \\ w_2 \\ \cdot \\ w_k \end{bmatrix} = \begin{bmatrix} R_1 & & & & U_1 F \\ & R_2 & & & U_2 F \\ & & \cdot & & \cdot \\ & & & R_k & U_k F \\ & & & & W_1 F \\ & & & & W_2 F \\ & & & & \cdot \\ & & & & W_k F \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_k \\ a^{DD} \end{bmatrix} + \begin{bmatrix} U_1 P_2^T \upsilon_1 \\ U_2 P_2^T \upsilon_2 \\ \cdot \\ U_k P_2^T \upsilon_k \\ W_1 P_2^T \upsilon_1 \\ W_2 P_2^T \upsilon_2 \\ \cdot \\ W_k P_2^T \upsilon_k \end{bmatrix}$$

[3]

QR Factorization

Note:

Need to 1^{st} solve equation below before solving for $x_1, ..., x_k$

$$\begin{bmatrix} w_1 \\ w_2 \\ . \\ w_k \end{bmatrix} = \begin{bmatrix} W_1 F \\ W_2 F \\ . \\ W_k F \end{bmatrix} a^{DD} + \begin{bmatrix} W_1 P_2^T \upsilon_1 \\ W_2 P_2^T \upsilon_2 \\ . \\ W_k P_2^T \upsilon_k \end{bmatrix}$$

[3]

Find LS Estimates

- a_k^{DD} is LS estimate of a^{DD}
- $x_{j|k}$, $j = 1, ..., k$ are LS estimates of x_j, $j = 1, ..., k$
- want to solve the following system:

$$R_j x_{j|k} = u_j - U_j F a_k^{DD}, \quad j = 1, ..., k$$

- need to find a_k^{DD}

[3]

More orthogonal transformations

Using a sequence of Householder transformations, need to solve the upper triangular system

$$S_k a_k^{DD} = \hat{w}_k$$

where S_k is nonsingular, upper triangular, and has $(m-1)$ rows

After a_k^{DD} is obtained, can solve for $x_{j|k}$ [3]

Other factors to consider:

- Computing the initial points
- Approximating the covariance matrices
- Fixing integer ambiguities
- Handling satellite rising and setting
    - not having a constant number of satellites

[3]

Dan Kalman.
An underdetermined linear system for gps.
The College Mathematics Journal, 33(5):384–390, November 2002.

Richard B. Thompson.
Global positioning system: The mathematics of gps receivers.
Mathematics Magazine, 71(4):260–269, October 1998.

Christopher C. Paige Xiao-Wen Chang.
An orthogonal transformation algorithm for gps positioning.
SIAM J. Sci. Comput., 24(5):1710–1732, 2003.