Continuous Wavelet Transform:

ECG Recognition Based on Phase and Modulus Representations

Edgar Gonzalez

Based on the paper by:
Lofti Senhadjii, Laurent Thoroval, and Guy Carrault [2]

May 12, 2009

Outline

Introduction

Biomedical signals:

- Fundamental to Analyzing Diseases
- Generally Time-Varying
- Non-stationary
- Usually Noisy

The Analyzing Tools:

- Fourier Transform
- Continuous Wavelet Transform (CWT)

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Electrocardiography (ECG)

ECG is the "recording of the electrical activity of the heart over time via skin electrodes." [1]

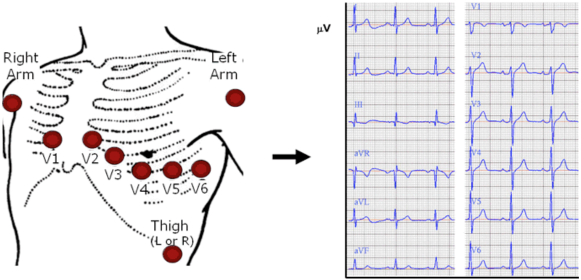

Fig. 1: Electrocardiogram and leads [1]

Facts of ECG:

- Voltage measured between pairs of electrodes
- Usually 12-Leads
- Diagnose a wide range of heart conditions
- and much more...

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Electrocardiograph

For a normal heart beat, the ECG usually looks like below:

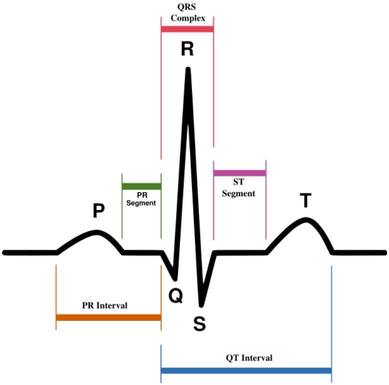

Fig. 2: Normal ECG [1]

Wavelets are used on ECG to:

- Enhance late potentials
- Reduce noise
- QRS detection
- Normal & abnormal beat recognition

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Layout of Presentation

1. Theoretical
   - CWT with complex analysis function
   - CWT square modulus (*scalogram*)
   - Local Symmetric Properties
2. See some examples

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The Continuous Wavelet Transform

$$(W_{\Psi} f)(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \cdot \overline{\Psi\left(\frac{t-b}{a}\right)}\, dt$$

- Ψ is complex, compactly supported, and hermitian ($\overline{\Psi}(t) = \Psi(-t)$)
- Ψ and f are at least twice continuous differentiable

The Square Modulus

We attempt to find an approximation to the square modulus. The square modulus of the CWT is defined as:

$$|(W_\Psi f)(a,b)|^2 = (W_\Psi f)(a,b)\overline{(W_\Psi f)(a,b)}$$

and

$$\frac{\partial |(W_\Psi f)(a,b)|^2}{\partial b} = \frac{\partial (W_\Psi f)(a,b)}{\partial b}\overline{(W_\Psi f)(a,b)} + (W_\Psi f)(a,b)\frac{\partial \overline{(W_\Psi f)(a,b)}}{\partial b}$$

A complex valued function $\Psi(x)$ can written as:

$$\Psi(x) = a(x) + ib(x)$$

and

$$\begin{aligned}\frac{d}{dx}\left[\Psi(x)\overline{\Psi(x)}\right] &= \frac{d}{dx}\left[(a(x)+ib(x))\cdot(a(x)-ib(x))\right] \\ &= \frac{d}{dx}\left[a^2(x)+b^2(x)\right] \\ &= 2(a(x)a'(x)+b(x)b'(x)) \\ &= 2\cdot \mathrm{Re}(\Psi'(x)\cdot\overline{\Psi(x)})\end{aligned}$$

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The derivative of $(W_\Psi f)$ with respect to b is:

$$\frac{\partial (W_\Psi f)(a,b)}{\partial b} = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \cdot \overline{\frac{\partial \Psi\left(\frac{t-b}{a}\right)}{\partial b}}\, dt$$
$$= \frac{1}{\sqrt{a^3}} \int_{-\infty}^{\infty} f(t) \cdot \overline{\Psi'\left(\frac{t-b}{a}\right)}\, dt$$

Using partial integration,

$$\frac{\partial (W_\Psi f)(a,b)}{\partial b} = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f'(t) \cdot \overline{\Psi\left(\frac{t-b}{a}\right)}\, dt$$

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Working out $\frac{\partial|(W_\Psi f)(a,b)|^2}{\partial b}$, we get

$$\frac{\partial|(W_\Psi f)(a,b)|^2}{\partial b} = 2\mathrm{Re}\left(\frac{\partial(W_\Psi f)(a,b)}{\partial b}\overline{(W_\Psi f)(a,b)}\right)$$

and using $\frac{\partial(W_\Psi f)(a,b)}{\partial b}$ above,

$$\frac{\partial|(W_\Psi f)(a,b)|^2}{\partial b} = \frac{2}{a}\mathrm{Re}\int_{-\infty}^{\infty} f'(t)\cdot\overline{\Psi\left(\frac{t-b}{a}\right)}\,dt$$

$$\int_{-\infty}^{\infty} f(t)\cdot\Psi\left(\frac{t-b}{a}\right)\,dt$$

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$$\int_{-\infty}^{\infty} f(t) \cdot \Psi\left(\frac{t-b}{a}\right)\, dt$$

For sufficiently smooth function $U(t)$ and small a (fine-scale),

$$\int_{-\infty}^{\infty} U(t) \cdot \overline{\Psi\left(\frac{t-b}{a}\right)}\, dt = a \int_{-\infty}^{\infty} U(ax+b) \cdot \overline{\Psi(x)}\, dx$$
$$\approx a^2 U'(b) \int_{-\infty}^{\infty} x \cdot \overline{\Psi(x)}\, dx$$

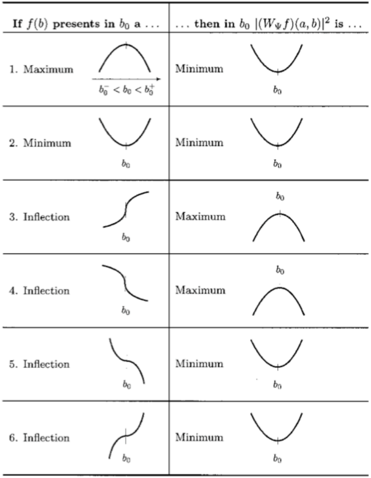
Applying the above approximation to $\frac{\partial|(W_\Psi f)(a,b)|^2}{\partial b}$,

$$\frac{\partial|(W_\Psi f)(a,b)|^2}{\partial b} \approx 2a^3 f'(b) \cdot f''(b) \cdot |m|^2$$

where

$$m = \int_{-\infty}^{\infty} x \cdot \overline{\Psi(x)}\, dx$$

Properties of approximation



If $f(b)$ presents in b_0 a ...	... then in b_0 $\|(W_\Psi f)(a,b)\|^2$ is ...
1. Maximum ($b_0^- < b_0 < b_0^+$)	Minimum (b_0)
2. Minimum (b_0)	Minimum (b_0)
3. Inflection (b_0)	Maximum (b_0)
4. Inflection (b_0)	Maximum (b_0)
5. Inflection (b_0)	Minimum (b_0)
6. Inflection (b_0)	Minimum (b_0)

Fig. 3: Properties of CWT vs. local symmetry of f [2]

Phase Behavior

By only considering the modulus of the CWT as above, the decomposed signal cannot in general be recovered.
We would need the phase to reconstruct.

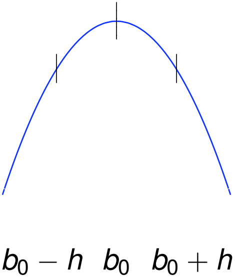
Locally Symmetric

Suppose f is continuous and satisfies the property below:

$$\exists b_0 \in \mathbb{R} \qquad \exists \epsilon > 0 \qquad \forall |h| < \epsilon$$

such that

$$f(b_0 + h) = f(b_0 - h)$$



For fine scale (small a)

$$
\begin{aligned}
(W_\Psi f)(a, b_0) &= \sqrt{a} \int_{-\infty}^{\infty} f(at + b_0) \cdot \overline{\Psi(t)}\, dt \\
&= \sqrt{a} \int_{0}^{\infty} (f(at + b_0) + f(-at + b_0)) \cdot \mathrm{Re}(\Psi(t))\, dt \\
&= 2\sqrt{a} \int_{0}^{\infty} f(at + b_0) \cdot \mathrm{Re}(\Psi(t))\, dt
\end{aligned}
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Then at a locally symmetric point of f, $(W_\Psi f)(a, b_0)$ is real and so the phase is 0 or π.

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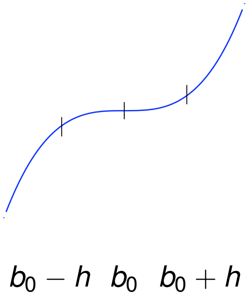
Locally Anti-Symmetric

Suppose f is continuous and satisfies the property below:

$$\exists b_0 \in \mathbb{R} \qquad \exists \epsilon > 0 \qquad \forall |h| < \epsilon$$

such that

$$f(b_0 + h) + f(b_0 - h) = 2f(b_0)$$



In a similar approach to the locally symmetric case,

$$\begin{aligned}(W_\Psi f)(a,b_0) &= \sqrt{a}\int_{-\infty}^{\infty} f(at+b_0)\cdot\overline{\Psi(t)}\,dt \\ &= -2i\sqrt{a}\int_{0}^{\infty} f(at+b_0)\cdot \mathrm{Im}(\Psi(t))\,dt\end{aligned}$$

Then CWT of f around b_0 is purely imaginary and the phase is $\pm\frac{\pi}{2}$.

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General Continuously Differentiable Function

Suppose now f is m times continuously differentiable function ($m \geq 2$)

$$(W_\Psi f)(a, b_0) \approx \sqrt{a} \cdot \sum_{n=1}^{m} \frac{a^n}{n!} \overline{m_n} f^{(n)}(b_0)$$

where

$$m_n = \int_{-\infty}^{\infty} x^n \cdot \Psi(x)\, dx$$

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Using only two terms

$$(W_\Psi f)(a, b_0) \approx -\sqrt{a^3} \cdot f'(b_0)\mathrm{Im}(m_1) \cdot i - \frac{\sqrt{a^5}}{2} f''(b_0)\mathrm{Re}(m_2)$$
$$= \alpha i + \beta$$

Notice the phase when $f'(b_0) = 0$ or $f''(b_0) = 0$.

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Some Examples

Let's look at a few examples:

1. Symmetric Properties
2. No Symmetric Properties
3. ECG

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Analyzing Wavelet

The mother wavelet for these examples:

$$\Psi(t) = g(t) \cdot e^{2i\pi k f_0 t}$$

where

$$g(t) = \begin{cases} C \cdot (1 + \cos(2\pi f_0 t)) & \text{for } |t| \leq \frac{1}{2f_0} \\ 0 & \text{elsewhere} \end{cases}$$

Analyzing Wavelet

$$\Psi(t) = g(t) \cdot e^{2i\pi k f_0 t}$$

For this wavelet:

- $k = 2$
- f_0 is the normalize frequency ($0 < f_0 < \frac{1}{2}$)
- $\mathrm{Im}(m_1)$ is positive
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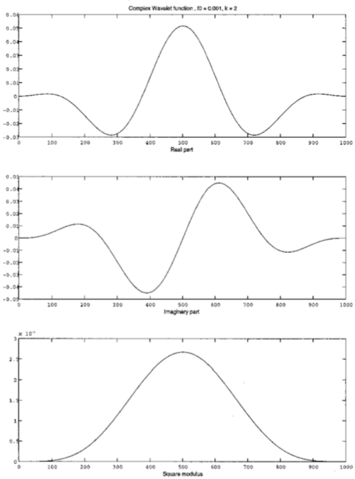

Fig. 4: Graph of Ψ [2]

1. Symmetrical Properties

For this example, $f_0 = 0.005$, and

$$a_i = \frac{f_0}{f_0 + i \cdot \Delta} \text{ with } \Delta = 0.005,\ 0 \leq i \leq 10$$

The input signal behaves like:

$$A_1 e^{-\frac{(t-m_1)^2}{b_1}} + A_2(t-m_2)e^{-\frac{(t-m_2)^2}{b_2}}$$

$$= 15e^{-\frac{(t-250)^2}{1700}} + 0.3(t-625)e^{-\frac{(t-625)^2}{2500}}$$

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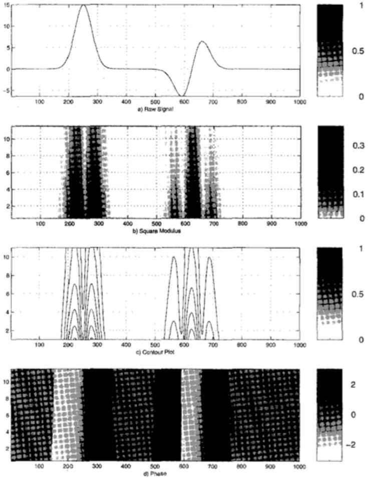

Fig. 5: Simulated data with symmetrical properties [2]

2. Without Symmetrical Properties

Define the following:

$$f_0(t) = \frac{1 - \exp(-\frac{(t-m_1)^2}{c})}{2 - \exp(-\frac{(t-m_2)^2}{c})} = \frac{1 - \exp(-\frac{(t-250)^2}{2500})}{2 - \exp(-\frac{(t-300)^2}{2500})}$$

2. Without Symmetrical Properties

The signal is defined by:

$$f(t) = 10000 \cdot f_0(t) + 225 \cdot f_0'(-(t+10))$$

In this case, the symmetry properties do not hold. But not all is lost...

$$\alpha i + \beta$$

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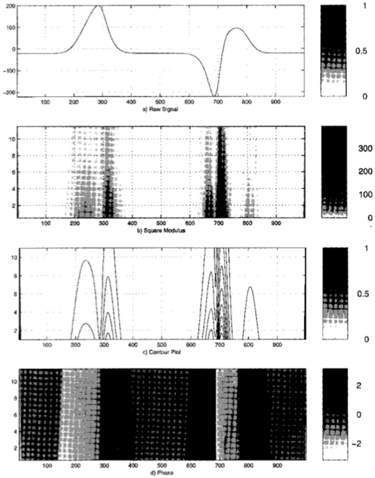

Fig. 6: Simulated data without symmetrical properties [2]

3. ECG Signal

The signal is a normal ECG sampled at 360 Hz with $f_0 = 0.001$, and

$$a_i = \frac{f_0}{f_0 + i \cdot \Delta} \text{ with } \Delta = 0.002,\ 0 \leq i \leq 25$$

3. ECG Signal

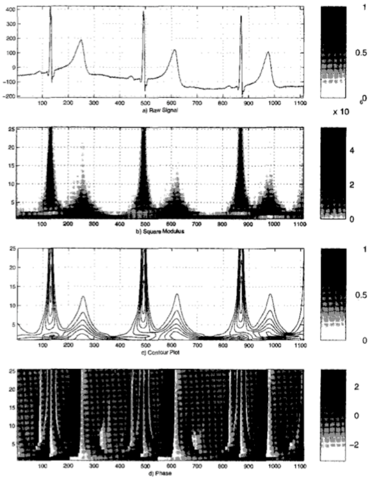

Fig. 7: ECG data [2]

3. ECG Signal

Notice that the P wave was not clearly separated from the QRS waves. One solution is a nonlinear transformation (NFT)

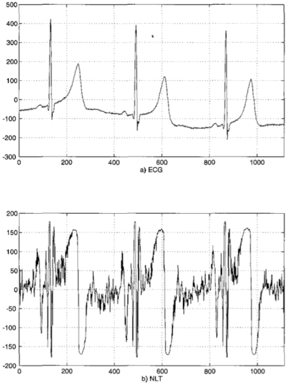

Fig. 8: NLT applied to ECG data [2]

3. ECG Signal

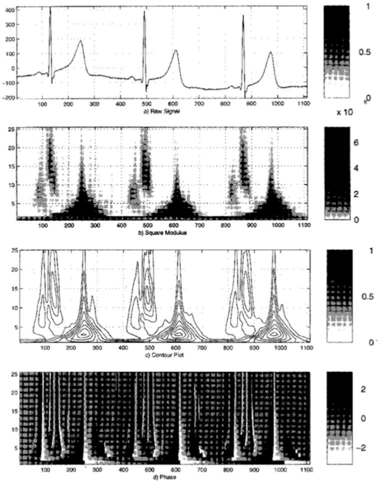

Fig. 9: CWT on NLT of ECG [2]

3. ECG Signal

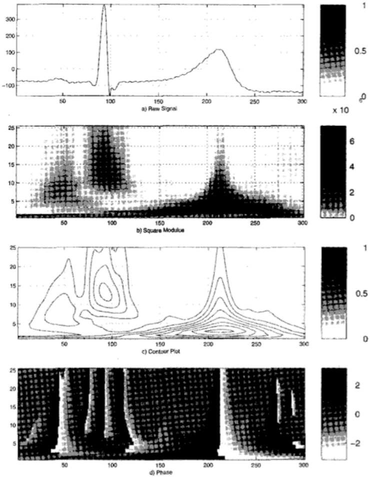

Fig. 10: Close up of CWT on NLT [2]

Conclusion

What was presented:

- Some theoretical properties of CWT
- Some examples including ECG
- The importance of phase

The above can be exploited for recognition signal processing via Markov Models.

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THE END

Wikipedia: Ecg, 2001.

Guy Garrualt Lofti Senhadjii, Laurent Thoraval.
Continuous wavelet transform: Ecg recognition based on phase and modulus representations.
University of Rennes, France.