Group #: ________ Members: ______________________________ Rating: ___________

1. (5 points) Given an *orthogonal* basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ for a vector space V, we can write any point $\mathbf{y} \in V$ as a linear combination of the basis elements with the formula $\mathbf{y} = \sum_{i=1}^{n} c_i \mathbf{b}_i$, where $c_i = \dfrac{\mathbf{y}^T \mathbf{b}_i}{\mathbf{b}_i^T \mathbf{b}_i}$. Notice that $c_i \mathbf{b}_i$ is essentially the *orthogonal projection* of $\mathbf{y}$ onto span$\{\mathbf{b}_i\}$. c_i is also known as the ith-coordinate of $\mathbf{y}$ relative to the basis $\mathcal{B}$.

   In this exercise, you will explore the differences in using different type of basis. Write $y = [1, 4, -2]^T$ as a linear combination in terms of the elements from the following bases.

   (a) (some random basis) $\mathcal{B}_1 = \left\{[1, 1, 1]^T, [2, 1, -2]^T, [-2, 1, 2]^T\right\}$

   (b) (an orthogonal basis) $\mathcal{B}_2 = \left\{[2, 2, 0]^T, [-1, 1, 0]^T, [0, 0, 4]^T\right\}$. (**no rref for (b) & (c)**)

   (c) (an orthonormal basis) $\mathcal{B}_3 = \left\{ \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]^T, \left[\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right]^T, \left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right]^T \right\}$.

   You must be wondering why in reality (c) is preferred over (b) which is then preferred over (a) since it seems like the algebra on (b) and (c) are more tedious. It turns out that doing (c) is least expensive in terms of computational time, since it requires $\sim 2n^2 - n$ flops (where n is the size of the vector) while (b) requires $\sim 4n^2 - 2n$ and (a) requires $\sim \frac{1}{3}n^3 + n^2 + O(n)$. (Computer uses flop [floating point operations: $+, -, \times, \div$] count to tell how expensive an algorithm is. The more flop an algorithm requires, the more expensive it is.)

2. (5 points) Given

$$\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}.$$

   Let $W = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and $S = \{\mathbf{u}_1, \mathbf{u}_2\}$.

   (a) (1 point) Determine whether S is an **orthogonal** basis for W. Justify your answer.

   (b) (3 points) (i) Find $\hat{\mathbf{y}} = \text{Proj}_W \mathbf{y}$ via $\sum_{i=1}^{2} \dfrac{\mathbf{y}^T \mathbf{u}_i}{\mathbf{u}_i^T \mathbf{u}_i} \mathbf{u}_i$. (ii) Compute $UU^T \mathbf{y}$, where $U = [\mathbf{u}_1 \quad \mathbf{u}_2]$. Notice that $UU^T \mathbf{y} \neq \text{Proj}_W \mathbf{y}$, when would we achieve equality in general?

   (c) (1 point) $\hat{\mathbf{y}}$ is the point in W that is closest to $\mathbf{y}$. We can determine whether $\mathbf{y} \in W$ by examining the distance between $\mathbf{y}$ and $\hat{\mathbf{y}}$. Find the distance from $\mathbf{y}$ to the plane in $\mathbb{R}^4$ spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$. Is $\mathbf{y} \in W$?