1. (5 points) Given an orthogonal basis $B = \{b_1, b_2, \ldots, b_n\}$ for a vector space $V$, we can write any point $y \in V$ as a linear combination of the basis elements with the formula

$$y = \sum_{i=1}^{n} c_i b_i,$$

where $c_i = \frac{y^T b_i}{b_i^T b_i}$. Notice that $c_i b_i$ is essentially the orthogonal projection of $y$ onto span$\{b_i\}$. $c_i$ is also known as the $i$th-coordinate of $y$ relative to the basis $B$.

In this exercise, you will explore the differences in using different type of basis. Write $y = [1, 4, -2]^T$ as a linear combination in terms of the elements from the following bases.

(a) (some random basis) $B_1 = \{[1, 1, 1]^T, [2, 1, -2]^T, [-2, 1, 2]^T\}$

(b) (an orthogonal basis) $B_2 = \{[2, 2, 0]^T, [-1, 1, 0]^T, [0, 0, 4]^T\}$. (no rref for (b) & (c))

(c) (an orthonormal basis) $B_3 = \left\{\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]^T, \left[\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right]^T, \left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right]^T\right\}$.

You must be wondering why in reality (c) is preferred over (b) which is then preferred over (a) since it seems like the algebra on (b) and (c) are more tedious. It turns out that doing (c) is least expensive in terms of computational time, since it requires $\sim 2n^2 - n$ flops (where $n$ is the size of the vector) while (b) requires $\sim 4n^2 - 2n$ and (a) requires $\sim \frac{1}{3}n^3 + n^2 + O(n)$. (Computer uses flop [floating point operations: $+,-,\times,\div$] count to tell how expensive an algorithm is. The more flop an algorithm requires, the more expensive it is.)

2. (5 points) Given

$$y = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \quad \text{and} \quad u_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}.$$

Let $W = \text{span}\{u_1, u_2\}$ and $S = \{u_1, u_2\}$.

(a) (1 point) Determine whether $S$ is an orthogonal basis for $W$. Justify your answer.

(b) (3 points) (i) Find $\hat{y} = \text{Proj}_W y$ via $\sum_{i=1}^{2} y^T u_i u_i$. (ii) Compute $UU^T y$, where $U = [u_1 \ u_2]$.

Notice that $UU^T y \neq \text{Proj}_W y$, when would we achieve equality in general?

(c) (1 point) $\hat{y}$ is the point in $W$ that is closest to $y$. We can determine whether $y \in W$ by examining the distance between $y$ and $\hat{y}$. Find the distance from $y$ to the plane in $\mathbb{R}^4$ spanned by $u_1$ and $u_2$. Is $y \in W$?