Testing for Graph Isomorphism Using Eigenvalues

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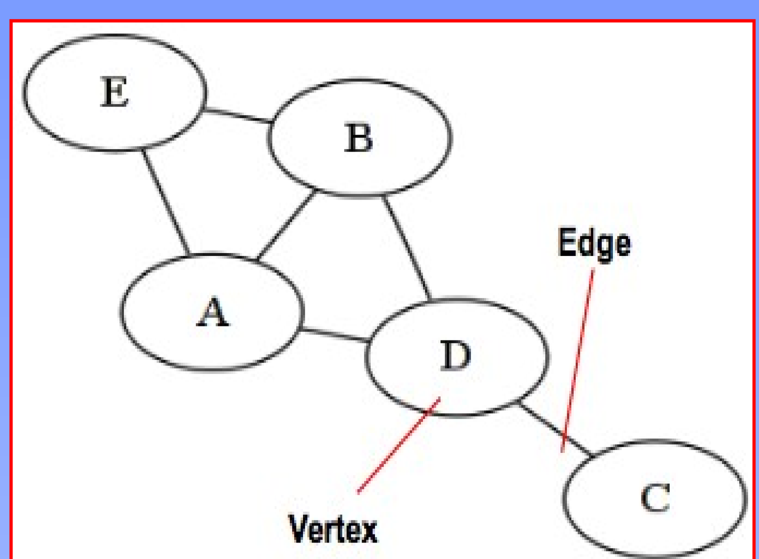
What is an Undirected Graph?

- The graphs we will be using are called undirected graphs, meaning that their points are unordered (Muhammad, "Definitions and Examples").
- The points of an undirected graph are called vertices. They are more like nodes than points.
- The lines connecting the vertices of an undirected graph are called edges.



Source: http://joaoff.com/category/education/

Figure 1. *This is an example of an undirected graph, with a vertex and an edge labeled.*

Graphs and Matrices

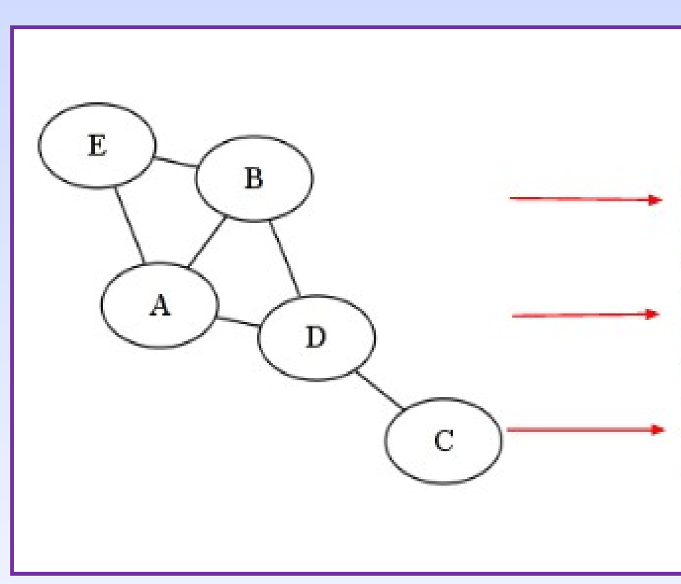
Converting Graphs into Matrix Form

Graph can be represented in matrix form by the following process:

1.Ensure that vertices of the graph are labelled.

2.Make a matrix of these labels, mapping each of the vertices to each other (as shown below).

3.For each pair of vertices, determine whether there is an edge connecting them or not. If there is, then put a 1; if not, then put a 0 (Muhammad, "Definitions and Examples").



	A	B	C	D	E
A	0	1	0	1	1
B	1	0	0	1	1
C	0	0	0	1	0
D	1	1	1	0	0
E	1	1	0	0	0

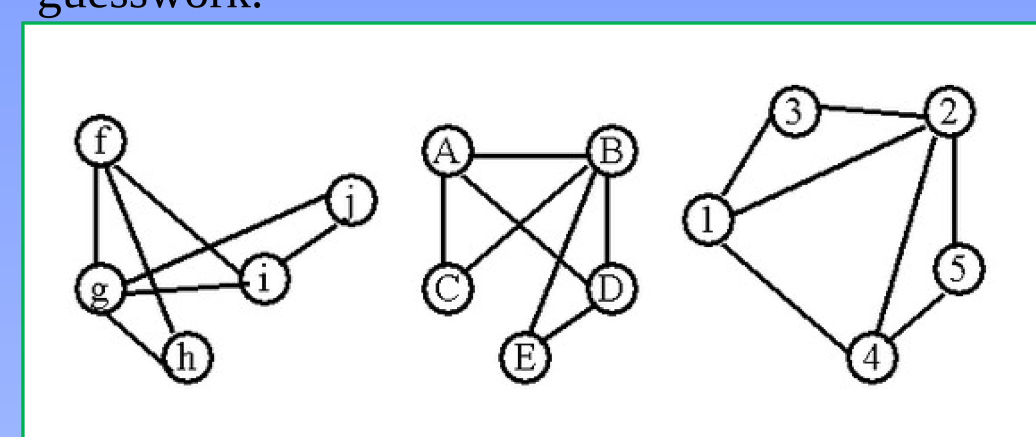
Source: http://joaoff.com/category/education/

Figure 2. *Example of a matrix representation of a graph.*

Graph Isomorphism

Graph isomorphism definition:

- Two graphs are isomorphic if their vertices can be rearranged and relabelled, without breaking any edges, to make the graphs identical (Adamchik, "Graph Theory").
- This means that isomorphic graphs, when relabelled correctly, should have exactly the same matrix representation.
- Problem is, we have no efficient way of knowing which labels should go where to produce this effect, aside from just pure guesswork.

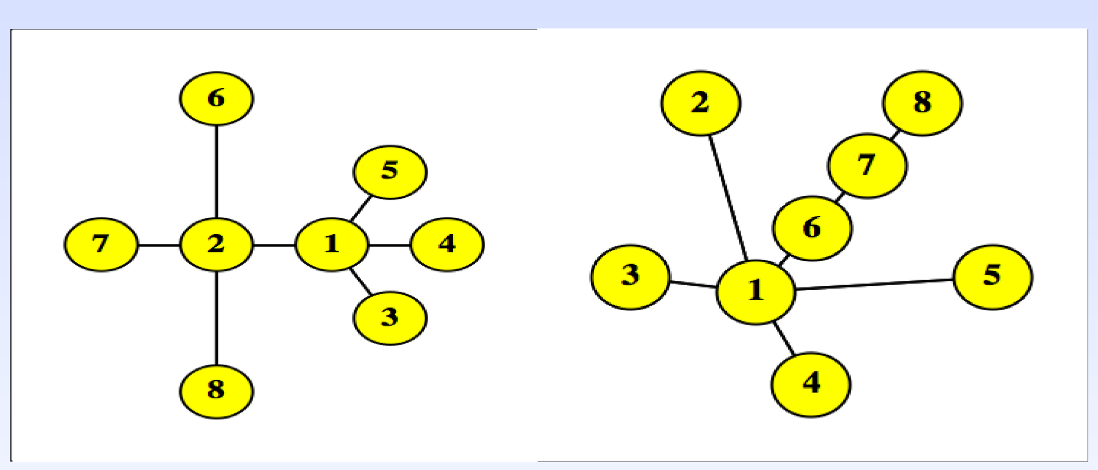


Source: http://www.cs.odu.edu/~cs772/fall04/lectures/public_key_cryptography.html

Figure 3. *Despite their appearances, the above graphs are isomorphic to one another.*

Eigenvalues and Non-Isomorphism

- While it is very difficult to prove that two graphs are isomorphic, it is relatively simple to prove that two graphs are non-isomorphic.
- Because two isomorphic graphs, when correctly relabelled have the same matrix representation, they have the same eigenvalues.
- However, not all graphs that have the same eigenvalues are necessarily isomorphic (Spielman, "Testing Isomorphism of Graphs with Distinct Eigenvalues").
- Thus, we cannot use eigenvalues to prove that two graphs are isomorphic.



Source: http://dabacon.org/pontiff/?p=4148

A B

Figure 4. *The above graphs are clearly non-isomorphic; they actually both have the same eigenvalue.*

A

$$\text{Eigenvalues}\left[\begin{pmatrix} 0&1&1&1&1&1&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&1&1\\ 0&0&0&0&0&1&0&1\\ 0&0&0&0&0&1&1&0 \end{pmatrix}\right]$$

B

$$\text{Eigenvalues}\left[\begin{pmatrix} 0&1&1&1&1&0&0&0\\ 1&0&0&0&0&1&1&1\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0 \end{pmatrix}\right]$$

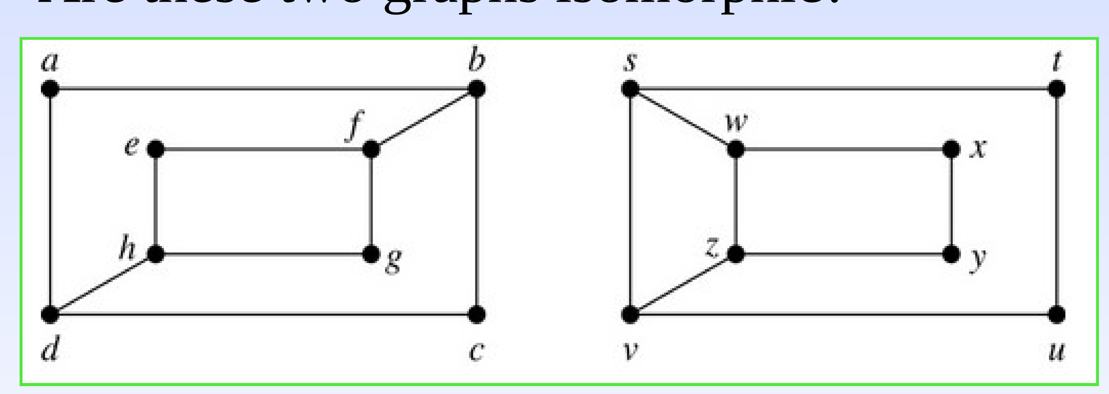
$$\left\{-\frac{1}{2}-\frac{\sqrt{13}}{2}, \frac{1}{2}+\frac{\sqrt{13}}{2}, \frac{1}{2}-\frac{\sqrt{13}}{2}, -\frac{1}{2}+\frac{\sqrt{13}}{2}, 0, 0, 0, 0\right\}$$

Proof

- However, we can use eigenvalues to show that two graphs are non-isomorphic.
- Claim: two graphs that have different eigenvalues cannot possibly be isomorphic
- Proof: Two isomorphic graphs can be rearranged and relabelled such that they both have the same matrix representation. Thus they have the same eigenvalues. Therefore, if two graphs do not have the same eigenvalues, then they cannot possibly be isomorphic ("Graph Isomorphism").
- Theorem: Similar matrices have the same characteristic polynomial and the same eigenvalues ("Graph Isomorphism").

Example

Are these two graphs isomorphic?



Source: http://www.cs.ucsb.edu/~pconrad/cs40/08S/notes/

A B

A

$$\text{Eigenvalues}\left[\begin{pmatrix} 0&1&0&1&0&0&0&0\\ 1&0&1&0&0&1&0&0\\ 0&1&0&1&0&0&0&0\\ 1&0&1&0&0&0&0&1\\ 0&0&0&0&0&1&0&1\\ 0&1&0&0&1&0&1&0\\ 0&0&0&0&0&1&0&1\\ 0&0&0&1&1&0&1&0 \end{pmatrix}\right]$$

- First we convert Graph A into this matrix representation.
- Second we compute its eigenvalue.

$$\left\{-\frac{1}{2}-\frac{\sqrt{17}}{2}, \frac{1}{2}+\frac{\sqrt{17}}{2}, \frac{1}{2}-\frac{\sqrt{17}}{2}, \frac{\sqrt{17}}{2}-\frac{1}{2}, -1, 1, 0, 0\right\}$$

B

$$\text{Eigenvalues}\left[\begin{pmatrix} 0&1&0&1&1&0&0&0\\ 1&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 1&0&1&0&0&0&0&1\\ 1&0&0&0&0&1&0&1\\ 0&0&0&0&1&0&1&0\\ 0&0&0&0&0&1&0&1\\ 0&0&0&1&1&0&1&0 \end{pmatrix}\right]$$

- Repeat the above process for B

$$\left\{-\frac{3}{2}-\frac{\sqrt{5}}{2}, \frac{3}{2}+\frac{\sqrt{5}}{2}, -\frac{1}{2}-\frac{\sqrt{5}}{2}, \frac{1}{2}+\frac{\sqrt{5}}{2}, \frac{1}{2}-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}-\frac{1}{2}, \frac{\sqrt{5}}{2}-\frac{3}{2}, \frac{3}{2}-\frac{\sqrt{5}}{2}\right\}$$

- Since the eigenvalues of A and B are clearly different, we can conclude that they are non-isomorphic.

Works Cited

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