The Leontief Production Model

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Leontief's Question

- In the 1930s, economist Wassily Leontief divided the economy into 500 sectors, writing linear equations that described how each sector distributed its output to other sectors. By creating this system of linear equations, he created a 500 by 500 matrix and began developing the *Leontief Input-Output (or Production) Model*.
- By creating this model, Leontief attempted to answer the following question: Is there a *production vector* (also known as the production level), ***x***, that equals the total demand for production in an economy? In other words, is there an ***x*** such that

$$\mathbf{x} = \mathbf{i} + \mathbf{d}$$

where ***i*** = (intermediate demand), and ***d*** = (external demand)

- There are two variations of this model—a closed and an open model. This presentation will deal with the *open* model with an initial ***d***.

- **Purpose**: *To find an **x** that satisfies both the internal demand and external demands of the hypothetical economy of "Nomansland."*

Answering Leontief's Question

Given the equation for the production vector

$$\mathbf{x} = \mathbf{i} + \mathbf{d},$$

we define the intermediate demand vector as the augmented consumption vectors times ***x***, or C***x***; thus, we can solve for ***x***:

$$\mathbf{x} = C\mathbf{x} + \mathbf{d} \rightarrow I_5\mathbf{x} - C\mathbf{x} = \mathbf{d} \rightarrow (I_5 - C)\mathbf{x} = \mathbf{d}.$$

This expression works because I_5-C can be inverted **So long as C contains non-negative entries and its column sums are less than 1, (I_5-C) is invertible**. A column sum less than 1 implies that not all production is being used within the economy of Nomansland and allows for an external demand of production (i.e., ***d***).

The Initial Conditions

- The Leontief Production Model makes three assumptions:
  - The prices of goods and services are held
  - input and output is measured in millions of dollars (hence referred to as "units of")
  - and the model assumes that for every sector of the economy there is a unit consumption vector ***c***, that lists the inputs needed for the units of output in the sector.

- Let *n* equal the number of sectors in the economy. For simplicity the following examples will hold *n* at 5—the five sectors of manufacturing, agriculture, communications, services (labor) and finances.

The Model

Assume for the economy of Nomansland, the units each sector consumes per unit of output are as follows (n = 5):

Inputs Purchased From...	Manufacturing (c_1)	Agriculture (c_2)	Communications (c_3)	Labor (c_4)	Finances (c_5)
Manufacturing:	.30	.15	.20	.15	.05
Agriculture:	.15	.35	.15	.35	.05
Communications:	.25	.05	.30	.20	.40
Labor:	.15	.25	.15	.10	.40
Finances:	.10	.15	.15	.10	.05

- If manufacturing were to produce 100 units, then it would consume 30 units from manufacturing; 15 from agriculture; 25 from communications; 15 from labor; and 10 from finances.
- The columns of the table the *consumption vectors* for each sector—denoted $\underline{\mathbf{c}}_j$. If x_j is the units produced by each sector, then $x_j\underline{\mathbf{c}}_j$ is the *intermediate demand* for the sector. Given the previous example:

$$x_1 = 100 \Rightarrow x_1\mathbf{c}_1 = 100\begin{bmatrix} .3 \\ .15 \\ .25 \\ .15 \\ .1 \end{bmatrix} = \begin{bmatrix} 30 \\ 15 \\ 25 \\ 15 \\ 10 \end{bmatrix}.$$

- In general, the total consumption of Nomansland is

$$\sum_{j=1}^{5} x_j\mathbf{c}_j = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3 + x_4\mathbf{c}_4 + x_5\mathbf{c}_5 = C\mathbf{x}.$$

Thus, for the economy of Nomansland,

$$I_5 - C = \begin{bmatrix} 1-.3 & -.15 & -.2 & -.15 & -.05 \\ -.15 & 1-.35 & -.15 & -.35 & -.05 \\ -.25 & -.05 & 1-.3 & -.2 & -.4 \\ -.15 & -.25 & -.15 & 1-.1 & -.4 \\ -.1 & -.15 & -.15 & -.1 & 1-.05 \end{bmatrix} = \begin{bmatrix} .7 & -.15 & -.2 & -.15 & -.05 \\ -.15 & .65 & -.15 & -.35 & -.05 \\ -.25 & -.05 & .7 & -.2 & -.4 \\ -.15 & -.25 & -.15 & .9 & -.4 \\ -.1 & -.15 & -.15 & -.1 & .95 \end{bmatrix}.$$

To find the inverse, augment (I_5-C) with I_5 and obtain the row-reduced echelon form; the right-side matrix will be $(I_5\text{-}C)^{-1}$:

$$\left[I_5 - C \mid I_5\right] \overset{rref}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 4.130 & 2.913 & 3.009 & 2.803 & 2.818 \\ 0 & 1 & 0 & 0 & 0 & 3.707 & 4.959 & 3.701 & 3.769 & 3.601 \\ 0 & 0 & 1 & 0 & 0 & 3.776 & 3.482 & 4.834 & 3.490 & 3.887 \\ 0 & 0 & 0 & 1 & 0 & 3.216 & 3.328 & 3.226 & 4.036 & 3.402 \\ 0 & 0 & 0 & 0 & 1 & 1.955 & 1.990 & 2.004 & 1.866 & 2.890 \end{bmatrix}.$$

Then, assume that the final demand vector ***d***—the production demanded by external sectors—is

$$\mathbf{d} = \begin{bmatrix} 250 \\ 350 \\ 200 \\ 300 \\ 150 \end{bmatrix}.$$

Finally, perform matrix multiplication to obtain ***x***:

$$\mathbf{x} = (I_5 - C)^{-1} \cdot \mathbf{d} = \begin{bmatrix} 4.130 & 2.913 & 3.009 & 2.803 & 2.818 \\ 3.707 & 4.959 & 3.701 & 3.769 & 3.601 \\ 3.776 & 3.482 & 4.834 & 3.490 & 3.887 \\ 3.216 & 3.328 & 3.226 & 4.036 & 3.402 \\ 1.955 & 1.990 & 2.004 & 1.866 & 2.890 \end{bmatrix} \begin{bmatrix} 250 \\ 350 \\ 200 \\ 300 \\ 150 \end{bmatrix}$$

$$\mathbf{x} \approx \begin{bmatrix} 3917.45 \\ 5073.45 \\ 4759.55 \\ 4335.10 \\ 2579.35 \end{bmatrix} \begin{matrix} \leftarrow \textit{Manufacturing} \\ \leftarrow \textit{Agriculture} \\ \leftarrow \textit{Communications} \\ \leftarrow \textit{Labor} \\ \leftarrow \textit{Finances} \end{matrix}$$

This is the production vector—the level of output at which sectors of the economy can satisfy their demand for inputs and the outputs of external entities. It is important to note that only at this production vector will the aforementioned demands be satisfied.

In fact, $(I_5 - C)\mathbf{x} = \mathbf{d}$ will always yield a unique production vector.

Is the Economy of Nomansland Productive?

- By finding a unique solution for ***x***, it is certain that Nomansland's economy is productive—satisfying its internal and external demand for production. However, productivity can also be generalized by analyzing *eigenvalues.*
- If A_{nxn} has a largest eigenvalue λ, then the corresponding eigenvalue of A^{-1} is λ^{-1}; then eigenvalue of $A_{nxn}+aI_n$ is $(\lambda+a)$, where *a* is any real number*;* lastly, since the eigen*vectors* of A and A^{-1} are the same, the eigenvalues of $(I_5\text{-}C)$ can be analyzed:

$$(I_5 - C) = -(C - 1I_5) : a = -1$$

$$\therefore eigenvalue_{(I_5 - C)} : -(\lambda - 1) = 1 - \lambda$$

$$\Rightarrow eigenvalue_{(I_5 - C)^{-1}} : \frac{1}{1-\lambda}$$

Consider $(I_5\text{-}C)^{-1}\underline{\boldsymbol{x}} = (1\text{-}\lambda)^{-1}\underline{\boldsymbol{x}}$:

1. If $\lambda < 1$, then $(I_5\text{-}C)^{-1}$ is invertible and thus the economy satisfies internal and external demand.
2. If $\lambda = 1$, then $(I_5\text{-}C)^{-1}$ is non-invertible and the economy is *closed.*
3. And if $\lambda > 1$, then $(I_5\text{-}C)^{-1}$ is not non-negative and thus the economy satisfies no demand.

Conclusion

- While 100-times smaller, this sampling of the Leontief Production Model gives insight into how the true, larger one functions: In the midst of simultaneous internal and external demand, there always exists a production level to satisfy them.
- The concepts in linear algebra and the study of economics are closely intertwined; applying vectors and matrices to economics is a concise way to organize economic schedules and make generalizations about an economy's productivity.
- The study of economics is profuse in today's society and it is important to organize, analyze and predict the state of things far into the future; the Leontief Production Model facilitates all three of these.

Acknowledgements

- http://home2.fvcc.edu/~dhicketh/LinearAlgebra/studentprojects/spring2006/nicholaskallem/Leontief%20project.htm
- http://www.math.dartmouth.edu/archive/m22f06/public_html/leontief_slides.pdf