The Linear Algebra Perspective of The Fibonacci Sequence and The Golden Ratio

Anthony Delarosa, Cassie Lewitzke, Tania Miller

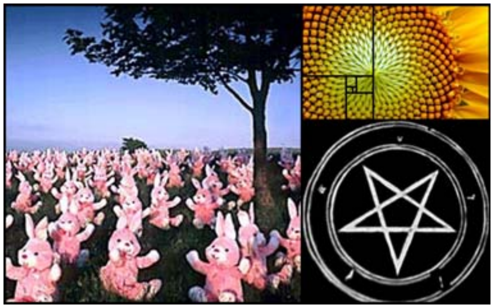
Department of Computer Science and Department of Mathematics and Statistics, California State University Long Beach, Long Beach, CA 90840

Introduction:

In about 540 B.C., A cult formed by the Greek philosopher Pythagoras of Samos, whose disciples went by the name of Pythagoreans, would use secret symbols to identify themselves as members, most notably the pentagram; a symbol that could only be correctly reproduced by Pythagoreans for only they knew the required ratio to create it. In 300 B.C., this ratio, now known as the golden ratio, was made public by its definition in book VI of Euclid's Elements. The golden ratio is widely perceived to be the ideal length to width proportions of nature, and some even believe it may contain mystical powers.

In 1202 A.D., A famous problem involving immortal incestuous rabbits was posed by the Italian mathematician Leonardo of Pisa, more commonly known as Fibonacci, in his historic book on arithmetic titled Liber Abaci, which translates to The Book of Abacus or The Book of Calculation; from this problem's solution a sequence of numbers was discovered, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ..., that is now commonly known as the Fibonacci sequence. These mystifying numbers can be found around nature in things such as the spiral arrangements of pinecones, the leaf arrangements in trees and sunflowers, among other things.

In 1609 A.D., it was discovered by Johannes Kepler, a German mathematician, that these two seemingly unrelated mathematical marvels do in fact share common ground. As it turns out, the Fibonacci sequence asymptotically approaches the golden ratio, a phenomenon easily proven using linear algebra.



Methods I:

Definition: An **eigenvector** of an $n \times n$ matrix A is a nonzero vector **x** such that $Ax = \lambda x$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution **x** of $Ax = \lambda x$; such an **x** is called an eigenvector corresponding to λ.

*The **eigenspace** consists of the zero vector and all the eigenvectors corresponding to λ.

Theorem:
The eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem:
If $\boldsymbol{v}_1, \dots, \boldsymbol{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_r\}$ is linearly independent.

* $\det A = \begin{cases} (-1)^r \cdot (\text{products of pivots in } U), & \text{when } A \text{ is invertible} \\ 0, & \text{when } A \text{ is not invertible} \end{cases}$

* The scalar equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of A.

* A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$$\det(A - \lambda I) = 0$$

Theorem: The Invertible Matrix Theorem (continued)
Let A be an $n \times n$ matrix. Then A is invertible if and only if:

s. The number 0 is *not* an eigenvalue of A.

t. The determinant of A is *not* zero.

Results:

Each term in the Fibonacci sequence equals the sum of the previous two. Creating a system of linear equations for the Fibonacci sequence:

$$\mathrm{F_n} = \mathrm{F_{n-1}} + \mathrm{F_{n-2}}$$
$$\mathrm{F_{n-1}} = \mathrm{F_{n-1}} + 0$$
$$\begin{bmatrix} \mathrm{F_n} \\ \mathrm{F_{n-1}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{F_{n-1}} \\ \mathrm{F_{n-2}} \end{bmatrix}$$

The general form for dynamic linear models like this is: $Au_0 = u_1$ where u_0 is the initial state, u_1 is the final state, and A is the transformation matrix that moves us from one state to the next. By the definition of eigenvalues and eigenvectors, we have the following identity:

$$AS = S\Lambda$$
$$A = S\Lambda S^{-1}$$
$$u_1 = S\Lambda S^{-1} u_0$$

This equation relates the initial state vector u_0 to the next state u_1. If we multiply A by k times. We get the following general form: $u_k = S\Lambda^k S^{-1} u_0$ (This is our linear model)

Now that we have a model, let's find the S, Λ and other parts that make it up. The first step is to find the eigenvalues and eigenvectors of the A matrix. This is given by: $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

To find the eigenvalues: $\det(A - \lambda I) = det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{bmatrix} = (1-\lambda)(-\lambda) - (1)(1) = 0$

$\lambda^2 - \lambda - 1 = 0$ then using quadratic equation $\lambda_1 = \frac{1+\sqrt{5}}{2}$ & $\lambda_2 = \frac{1-\sqrt{5}}{2}$

For $\lambda_1 = \frac{1+\sqrt{5}}{2}$: $(A - \lambda_1 I)x = 0 \leftrightarrow \begin{bmatrix} 1-\left(\frac{1+\sqrt{5}}{2}\right) & 1 \\ 1 & -\left(\frac{1+\sqrt{5}}{2}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

X_2 is our free variable. Therefore: $\left(\frac{1-\sqrt{5}}{2}\right) x_1 + 1 = 0 \leftrightarrow x_1 = \frac{2}{\sqrt{5}-1}$

Using the old algebra trick for the difference of squares, we can simplify this

$$x_1 = \frac{2}{\sqrt{5}-1} \cdot \frac{(\sqrt{5}+1)}{(\sqrt{5}+1)} = \frac{2(\sqrt{5}+1)}{5-1} = \frac{\sqrt{5}+1}{2}$$

Our first eigenvector: $v_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$

Then we do the same process for finding the eigenvector for λ_2 which leads you to $v_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$

Note that the vectors v_1 and v_2 are orthogonal to each other. (Using orthogonal theorem)

Now that we have the eigenvalues and eigenvectors, we can write the S and Λ matrices as follows: $S = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$ & $\Lambda = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}$

By using the Inverse matrix formula we get: $S^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{\sqrt{5}-1}{2} \\ 1 & \frac{\sqrt{5}+1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{\sqrt{5}-1}{2\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{\sqrt{5}+1}{2\sqrt{5}} \end{bmatrix}$

$u_0 = Sc$ Where c is a 2 x 1 vector of scalars. $c = S^{-1} u_0$

Putting everything together, we can write our final model for the Fibonacci sequence:

$$u_k = S\Lambda^k c$$

$$\begin{bmatrix} F_{k+3} \\ F_{k+2} \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^k \left(\frac{5+\sqrt{5}}{10}\right) + \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \left(\frac{1-\sqrt{5}}{2}\right)^k \left(\frac{5-\sqrt{5}}{10}\right)$$

This equation gives the k+3 and k+2 terms of the Fibonacci sequence as a function of just one variable: k. This allows us to easily find any term we'd like -- just plug in k.

The Fibonacci Sequence is defined recursively using the following formula:

$$F_n = F_{n-1} + F_{n-2}$$

Therefore, using this formula we need to calculate the two prior numbers, n-1 and n-2, in order to calculate n. On the other hand, with the application of Linear Algebra, as demonstrated above, an explicit formula is created to find any number in the Fibonacci Sequence. For example, the Fibonacci Sequence is as follows:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

To demonstrate that the explicit formula is accurate, we will use it to find $F_{11} = 55$.

Since we want to find the 11th term we will let k = 8

$$\begin{bmatrix} F_{k+3} \\ F_{k+2} \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^k \left(\frac{5+\sqrt{5}}{10}\right) + \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \left(\frac{1-\sqrt{5}}{2}\right)^k \left(\frac{5-\sqrt{5}}{10}\right)$$

$$\begin{bmatrix} F_{8+3} \\ F_{8+2} \end{bmatrix} = \begin{bmatrix} F_{11} \\ F_{10} \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^8 \left(\frac{5+\sqrt{5}}{10}\right) + \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \left(\frac{1-\sqrt{5}}{2}\right)^8 \left(\frac{5-\sqrt{5}}{10}\right)$$

$$\begin{bmatrix} F_{11} \\ F_{10} \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} 33.9941 + \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} .0059 \qquad \begin{bmatrix} F_{11} \\ F_{10} \end{bmatrix} = \begin{bmatrix} 55 \\ 34 \end{bmatrix}$$

Methods II:

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

*A is a diagonalizable if an only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$. We call such a basis an **eigenvector basis.**

Theorem:
An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

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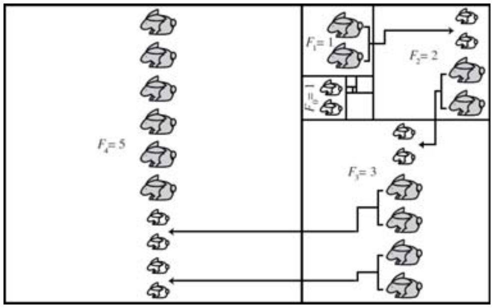
Conclusion:

Our linear model gives us a function for which we can calculate the n^{th} term in the Fibonacci sequence. As we can see in the diagram below, our function approaches the golden ratio as n increases to infinity.



Summary:

With a little linear algebra we have unlocked the secrets of the Pythagoreans and the immortal rabbits. We have demonstrated how a simple linear model can break down the complexities of the mysterious Fibonacci sequence into a function of just a single variable. Now should we ever encounter a real world situation of immortal incestuous rabbits, we would know the exact date they would exceed our population and take over the world.

You have also seen how this function approaches a constant value as its single variable gets bigger and bigger; what's interesting of course is how this constant value is the same ratio used by Pythagoras and his disciples to pass their esoteric messages in the form of pentagrams over two thousand years ago. One can only imagine what other secrets remain to be unlocked by the all seeing eye of linear algebra.

Acknowledgements:

Andre Chamberlain, Dr. Darin Goldstein, H. Peter Aleff, Dr. Jen-Mei Chang, Mario Livio, K.K. Tung, and Richard A. Dunlap. *Picture Citations:* http://bw033.k12.sd.us/Textbook%20clip%20art.gif , http://cowshell.com/uploads/drawergeeks/bunnies.jpg, http://www.disclose.tv/files/photos/2ba61cc3a8f4414L.jpg,http://www.knowledgerush.com/wiki_image/c/c4/Pentagram4.png, http://img.domaintools.com/blog/island-bunnies.gif, http://press.princeton.edu/chapters/s8446.pdf